

2002

# Characterization of DEA ranking models

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**Characterization of DEA ranking models**

by

**Sung-Kyun Choi**

A dissertation submitted to the graduate faculty  
in partial fulfillment of the requirements for the degree of  
**DOCTOR OF PHILOSOPHY**

**Major: Industrial Engineering**

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**2002**

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## **ACKNOWLEDGEMENTS**

First of all, I would like to express my sincere appreciation and thanks to my major professor, Dr. Tim Van Voorhis, for his advice, guidance, and patience through all phases of my graduate study and the preparation of this dissertation.

I would also like to thank Dr. Kenneth Koehler, Dr. K. Jo Min, Dr. Sarah Ryan and Dr. Sigurdur Olafsson for their helpful comments, feedbacks, and suggestions regarding my research as well as their participation on my committee.

I would like to thank Korean National Railroad (KNR), Ministry of Government Administration and Home Affairs (MOGAHA) for their financial supports through my graduate study.

My warmest thanks go to my parents, Kang-Hee Choi and Yong-Sook Suh, and my wife, Hye-Young Na. Without their endless love, understanding and encouragement, I could not have finished my graduate study.

## ABSTRACT

Other than measuring relative efficiency, DEA (Data Envelopment Analysis) has been used in a number of other ways to elaborate further on the performance of individual units or to ascertain how the units could become more efficient. Also researchers have developed methods for using DEA as a ranking model.

We classified DEA ranking models into two categories based on whether preferences (weights) are given or not. When the decision maker's preferences (weights) are not given, the ranking criteria and corresponding ranking results of each model vary depending on the methods each model uses. When the decision maker's preferences (weights) are given, the accuracy and acceptability of the results depend on how well these given preferences are reflected to each weight restriction method.

Since the ranking result from each model is determined by the characteristics each model has, it is important to understand these characteristics. This hopefully can help decision makers to make a better decision. In this dissertation, we analyze the characteristics of A-P (Andersen-Peterson) model and cross-efficiency evaluation in category 1, and cone-ratio and Wong and Beasley weight restrictions in category 2. Alternative models for measuring overall efficiency are proposed.

To better characterize ranking models, we define a new metric, the specialization index (SI), and propose using the  $A_i$  score in cross-efficiency evaluation to identify specialized DMUs. Also we examine the popular characterization on the 1<sup>st</sup> ranker of cross-efficiency evaluation and show that it is not always true. The fixed weighting nature of cross-efficiency evaluation is analyzed in the multiple-input, multiple-output situation analytically and

empirically. Biplots are proposed as a method for comparing the characteristics of model with multiple inputs and/or outputs visually.

On the characteristics of cone-ratio weight restrictions, we suggest two properties (P1) and (P2). Property (P1) shows a way to measure the efficiency score when cone-ratio weight restrictions are applied under constant returns to scale with single- input, multiple-outputs. Based on this property, we propose some graphical explanations of other DEA issues.

We investigate the characteristics of Wong and Beasley weight restrictions and compare both their theoretical implications and empirical behavior with those of cone-ratio weight restrictions. We show that under Wong and Beasley weight restrictions, each DMU takes all different weight vectors and some DMUs may have limiting efficiency score.

Finally, we present alternative models for measuring each of overall efficiency (OE) with cone-ratio weight restrictions and compare with previous models using examples.



## **CHAPTER 1. INTRODUCTION**

### **1.1 Introduction**

Data Envelopment Analysis (DEA) is a method to determine the relative efficiencies of a set of organizational units such as schools, hospital departments or bank branches when there are multiple incommensurate inputs and outputs [8]. Measuring relative efficiencies of the units has been obviously the first purpose of DEA assessment to enhance various notions of efficiency.

However as indicated in [7], [38], DEA can be used in a number of other ways to elaborate further on the performance of individual units and to ascertain how the units become more efficient. Some of the further uses are 1) Identifying peer groups, 2) Identifying efficient operating practices (or efficient strategies), 3) Target setting, 4) Monitoring efficiency changes over time, 5) Resource allocation, etc. The type of information derived from assessment of performance depends on the aim of the assessment and on the particular assessment method used. The applicability and practicability of DEA can be easily confirmed by a recent survey, which has compiled more than 1,000 previous research efforts [28].

Also researchers have developed methods for using DEA as a ranking model, which results in a recent review by Adler et al. [1]. In this dissertation, we combine the previous two recent reviews on DEA ranking models, Adler et al. [1] and Allen et al. [2], and classify DEA ranking models into two categories according to whether preferences (weights) are given or not.

In fact, many DEA ranking models start with the assumption that there are no given preferences (criteria), which is often the case in real life applications. Since, many DEA ranking models have been developed mainly for overcoming the weakness of CCR (Charnes,

Cooper and Rhodes) model [9], - i.e. producing many efficient DMUs-, and increasing the discriminating power of traditional CCR model using some specified methods, the ranking criteria and corresponding results of each model vary by the methods each model used. Therefore, when a decision maker makes a decision based on a certain single DEA ranking model without clear understanding of the characteristics of each ranking model, it may results in a misleading decision. For example, a decision maker can try A-P (Andersen-Petersen) model [3] or cross-efficiency evaluation [13], [14] to make a ranking for a certain management decision. However, these two models often make far different ranking results and sometimes make similar ranking results. And these make decision makers difficult to have confidence in their final decisions.

On the other hand, in case that decision maker's preferences (weights) are given, the accuracy and acceptability of the results depend on how well these given preferences are reflected to each weight restriction method. In DEA, decision maker's preferences (weights) on variables are often reflected as cone-ratio or Wong and Beasley [41] weight restrictions. The cone-ratio (C/R) weight restrictions interpret decision maker's preferences (weights) as relative importance of input (output) and the Wong and Beasley (W/B) weight restrictions interpret those as virtual proportion of each input (output) to total virtual input (output). However, the characteristics of W/B weight restriction have not sufficiently explored and there has been no attempt to compare the characteristics of W/B weight restrictions with C/R weight restrictions.

And these motivate this research on the characteristics of each DEA ranking model, which hopefully can help decision makers to avoid misleading decisions and can provide more information for their decisions.

The rest of this chapter is organized as follows. In section 1.2, general assumptions of DEA and two different views on the characteristics of the CCR model are described. In section 1.3, we review the current discussions on the DEA ranking models. This consists of 3 subsections such that discussions on the comparison of DEA and MCDM, classification of DEA ranking models and some critical views on DEA ranking models. In this section, we suggest the classification of DEA ranking models as of two categories. In section 1.4, we described that the purpose of using DEA ranking models can go beyond ranking to importing additional useful information. Finally, the organization of this dissertation is provided in section 1.5.

## **1.2 The characteristics of the CCR model**

### **1.2.1 The general assumptions of DEA**

The practical application of DEA presents a range of procedural issues to be examined and resolved. Recently Dyson et al. [15] presented some of the pitfalls and protocols in application of DEA under the general assumptions as follows.

#### **(1) Homogeneity assumption**

- a) The units are undertaking similar activities and producing comparable products or services so that common set of outputs can be defined, b) A similar range of resources is available to all the units, c) The units are operating in similar environments, since the external environment generally impacts upon the overall performance of units.

#### **(2) Assumptions on input / output set**

- a) It covers the full range of resources used, b) It captures all activity levels and performance measures, c) The set of factors are common to all units, d) The

c) environmental variation has been assessed and captured if necessary

(3) Assumptions on factor measurement

a) The inputs and outputs are isotonic, i.e. increased input reduces efficiency, whilst increased output increases efficiency, b) Measurement scales of the inputs and outputs should conform to ratio scales

(4) Linearity assumption

a) The weights are the assigned values or prices of the inputs and outputs and coupled with the ratio scales of the factors imply linear value functions, b) However, this linearity may be problematic, as for some outputs, the value of additional output may begin to diminish.

### 1.2.2 Two different views on the characteristics of CCR model

There exist two somewhat different views on the characteristics of the CCR model and the following critical view (1) is directly related to DEA ranking models. We believe that it is necessary to research further on the characteristics of DEA ranking models to mitigate these critical views and to expand the availability of using DEA ranking models.

(1) The view that indicates the problems of DEA

Since the CCR model places no constraints on weights (except for positivity), a DMU (Decision Making Unit) that is superior to all other units in any single output / input ratio can be evaluated as technically efficient. That is, the CCR model will assign high weights to the inputs and outputs for which the DMU is particularly efficient and low weights (including zero weights) to all the other inputs or outputs. Therefore the CCR model results in many technically efficient DMUs especially when there are

relatively many input, output variables compared to the number of DMUs.

Many researchers pointed out the problems of DEA, such as 1) it produces many efficient DMUs and therefore cannot discriminate between all DMUs since a DMU, which specializes in one task, will be evaluated as efficient, 2) if the assigned weights don't reflect the decision maker's preference (judgment) or are extremely biased, the results may not be accepted by the decision maker.

(2) The view that asserts the flexibility in choosing weights is one of the major advantages of DEA.

In much of the DEA literature, increased flexibility in choosing weights is considered to be one of DEA's major advantages when compared to other techniques used to measure efficiency.

### **1.3 Review of discussions on the DEA ranking models**

#### **1.3.1 DEA and multi-criteria decision-making (MCDM)**

DEA arises from situations where the goal is to determine the productive efficiency of a system or DMU by comparing how well these units convert inputs into outputs, while MCDM models have arisen from problems of ranking and selecting from a set of alternatives that have conflicting criteria [35]. Therefore the MCDM literature was entirely separate from DEA research until 1988, when Golany combined interactive, multiple-objective linear programming in DEA [1].

Stewart [34], [35] has compared the traditional goals of DEA and MCDM and indicated on the philosophical distinction between two methods such that while MCDM is generally concerned with the coherent elicitation of human value judgment, DEA tends to avoid

inserting human value judgments, aiming more at the objective data.

On the other hand, some researchers, Doyle et al. [12], Sarkis [26] and Khouja [19], showed a methodological connection between DEA and MCDM with replacing the minimization criteria of MCDM as inputs in DEA and maximization criteria of MCDM as outputs in DEA. For example, Doyle et al. showed a methodological connection using an example, “choice between alternative sites for an electric power plant in different European countries”, which was explained by Stewart [33] using an MCDM.

However, it should be noted that certain researchers have argued that MCDM and DEA are two entirely separate approaches, which do not overlap. MCDM is generally applied to *ex ante* problem areas where data are not readily available, especially if referring to a discussion of future technologies, which do not yet exist. DEA, on the other hand, provides an *ex post* analysis of the past from which to learn [6].

### 1.3.2 Classification of DEA ranking models

Most recently, Adler et al. [1] reviewed previously published DEA ranking methods by dividing them into six categories.

(1) Cross-efficiency evaluation (2) A-P method (3) Methods based on benchmarking (4) Methods utilizing multivariate statistical techniques (5) Methods of ranking inefficient units (6) Methods which require the collection of additional, preferential information from relevant decision-makers and combine MCDM methodologies.

Among the above 6 areas, only (6) requires subjective information, i.e., the human value judgment on weights. All the other 5 areas are purely based on the objective input, output values. Even though Adler et al. [1] mentioned methods incorporating human value

judgment in DEA, they focused more on the combined method with MCDM but they didn't fully describe another important stream of the DEA methods, which also incorporates human value judgments directly into DEA formulation.

In respect to this research area, Allen et al. [2] provided a detailed review with subtitle that 'evolution, development and future research directions on the weight restrictions and value judgments in DEA'.

Combining the above two research works [1] and [2], in this paper we classify the ranking methods in two categories whether it applies the given weight restrictions (preferences) to all DMUs or not.

(1) Ranking methods in DEA without given weight restrictions (preferences)

DEA ranking models in this category are still can be divided into two groups whether they apply common weight restrictions to all DMUs or not. Above (1) ~ (5) areas fall in this category. The A-P model uses different weights to each DMU. But cross-efficiency model is revealed to use almost common (fixed) weights by [4] and this research (details are described in chapter 3 and 4).

(2) Ranking methods in DEA with given weight restrictions (preferences)

The following three methods correspond in this category. a) direct weight restriction (Assurance region methods) : Type I, II and absolute weight restrictions, b) restricting virtual inputs and outputs (Wong and Beasley method), c) adjusting the observed input-output levels.

The models in category (2) can be thought of as using the same criteria to all DMUs and thus the efficiency score of each DMU is measured under the same criteria. When we can get the information on the preferences of decision maker, these common criteria can be

directly made based on those preferences, however when we cannot get the information on those, the weights can be made according to each model's own way.

As a ranking model, MCDM seems to use the common criteria for making the choice between alternatives. When we use a ranking model in the narrow sense, it would be reasonable to use a certain MCDM method or DEA ranking model with common weight restrictions since the model should reflect the preference (criteria) of one decision maker or the aggregated preferences of many decision makers.

On the other hand, many other DEA ranking models have been developed mainly for overcoming the weakness of CCR model, - i.e. producing many efficient DMUs-, and increasing the discriminating power of traditional CCR model using some specified methods. And with this increased discrimination, each model can find the ranking by the efficiency score.

### **1.3.3 Some critical views on the DEA ranking models**

The DEA efficiency score of each DMU is measured by maximizing the ratio of virtual output / virtual input, and therefore we let the variables that have minimization criteria as inputs and those those have maximization criteria as outputs.

Unless we have weight restrictions on input or output variables obtained by human value judgments, DEA ranking methods should find another ranking criterion that can restrict or adjust the weights appropriately to make a full ranking.

Methods which are classified in previous section as 1) Ranking methods in DEA without given weight restrictions correspond to this case, and they find the weights in each of its own way. For example,



- 1) The A-P model finds the weights by comparing the unit under evaluation with a linear combination of all other units in the sample in which the unit being evaluated is excluded. Therefore the efficiency score from A-P model represents the maximum proportional increase in inputs (decrease in outputs) preserving efficiency.
- 2) The cross-efficiency model finds the weights by minimizing the summation of all the other DMU's virtual output subject to the following constraints. a) The summation of all the other DMU's virtual input is equal to 1, b) The efficiency score of all the other DMUs can not exceed 1 while keeping the CCR-efficiency score of DMU being evaluated.
- 3) Friedman et al [17] and Sinuany-Stern et al [32] presented two ranking methods using canonical correlation analysis (CCA / DEA) to find a single set of weights to all DMUs and linear discriminant analysis (LDA) to find a score function value, given the result of DEA, which shows the division of DMUs into efficient and inefficient.

However no matter which method each model uses to find weights, the applied weights are severely dependent on the sample data and the method to find weights. The applied weights on each variable under these methods might be far from the decision-maker's opinion in some cases and then it would be hard to be accepted.

As indicated by Pedraja-Chaparro [22], total flexibility, which is an implicit assumption of the above model, has been criticized for several aspects.

- 1) Factors of secondary importance may dominate a DMU' efficiency assessment or important factors may be all but ignored in the analysis.
- 2) The implicit assumption, which allows weight flexibility in DEA, is that each DMU may have individual objectives and particular circumstances. However considering the general assumption of DEA, i.e., evaluating homogeneous DMUs using the same input

and output variables under the same overall objective, it may not be acceptable when the weights attached to each input and output variable are greatly different.

3) Finally, a certain amount of information regarding the importance of inputs and outputs might be available in some cases.

On the other hand, each method classified in previous section as 2) Ranking methods in DEA with given weight restrictions tries to incorporate the value judgment of the decision maker or expert as the applied weights to input, output variables. When the result is obtained based on these suggested weights, it would presumably be readily accepted by the decision maker.

#### **1.4 Considerations on the purposes of DEA ranking models**

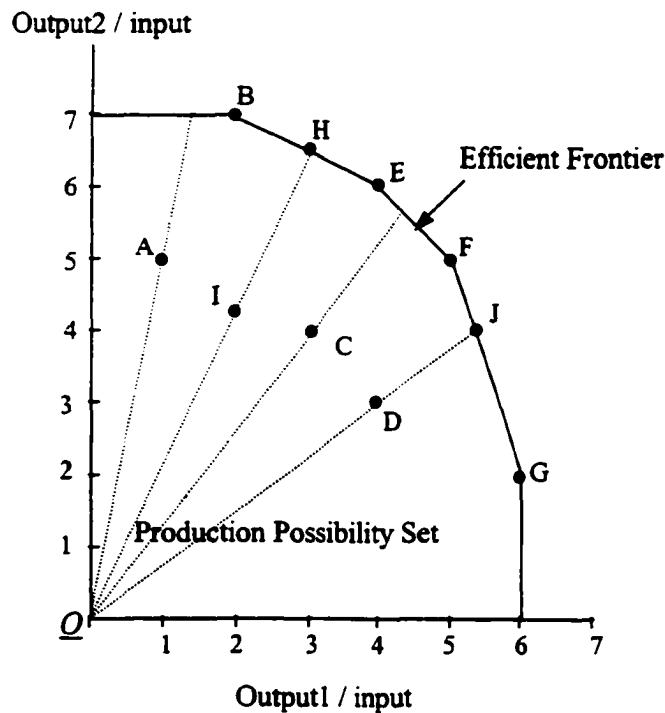
As we described in previous section, a lot of research has been done to make a whole ranking among DMUs (more specifically to differentiate CCR-efficient DMUs) along with the critical views. However, in spite of these research efforts on DEA ranking models, we cannot easily find the descriptions about the purposes of DEA ranking models and their corresponding ranking criteria. Therefore in many DEA ranking models, we often feel difficulty in explaining the purpose or corresponding ranking criteria except the fact that using those criteria, it can make a ranking among whole DMUs.

The fact that each DMU can have the flexibility in choosing weights in its most favorable light has been considered as one of the most valuable characteristics in DEA. Based on this flexibility, DEA tries to find the information most appropriate to each DMU rather than trying to find the general information among all DMUs in the sample.

Therefore the purpose of many DEA ranking models based on this flexibility, i.e., many

of DEA ranking models which don't apply common weight restriction, is more often to give some useful information to each DMU, rather than to give general information among all DMUs.

For example in Figure 1.1, DMU I is compared with DMU H in measuring efficiency, but not compared with DMUs F or G. On the other hand, DMU D is compared with DMU J in measuring efficiency, but not compared with DMUs B or E. That is, DMU I and DMU D are evaluated by the different criteria. The weight multipliers of DMU I are automatically chosen in DEA to achieve the highest CCR efficiency score.



**Figure 1.1 One input and two outputs case**

Therefore using DEA, the decision maker of DMU I can get the information on the current CCR efficiency score, peer groups, target points, frontier DMUs, etc. Decision makers of each DMU can get the similar information using DEA, which has been considered

to be the traditional purposes of using DEA as briefly explained in section 1.1.1.

We think that the most of the controversy on DEA ranking models comes from the misunderstanding of the purpose of each ranking model or omission of defining the purpose of each ranking models. Therefore, in this thesis we define the purposes of using DEA ranking models as of two kinds.

- (1) (Narrow sense) To make a choice between alternatives (selection) by the given preference (criteria) of one decision maker or by single aggregated preference of many decision makers.
- (2) (Traditional sense) To find more information corresponding to the general purpose of DEA to each DMU. For example, a decision maker can get information on ranking, peer groups as well as the changes of efficiency scores of sample DMUs under each model's criteria, which can be reflected to the management decisions.

In the previous section, we classified the DEA ranking models in two categories and we also classified the models in category (1) into two groups whether it applies the common criteria (common weight restrictions) to all DMUs or not. We also classified the purpose of using DEA ranking models as of two kinds above. The reason of this classification is to clarify the relationship of the purpose and the choice of DEA ranking models. In fact, the purpose of using DEA ranking models (even though they have the name of ranking model) doesn't lie solely in definition (1) or (2) above.

Many DEA ranking models are based on rather different assumptions than those used for MCDM ranking models. That is, many DEA ranking models start from the assumption that there are no given preferences (criteria), which is often the case in real life applications,

and therefore the ranking criteria of each model varies by the methods each model uses. Even though they may be criticized when using for the purpose of narrow sense such that 1) each DMU is evaluated by different criteria 2) since each DMU is evaluated by different criteria, it is problematic to make a unique ranking among all DMUs, they still can be useful for the purpose of traditional sense.

On the other hand, when we have the information on preferences of one decision maker or the aggregated preferences of many decision makers that can be represented as common weighting scheme in DEA, a certain DEA ranking model can be used for the purpose of narrow sense (1) without the least dissatisfaction.

Although DEA ranking models can be used for the above two purposes of (1) and (2), it is necessary to develop more precise weighting schemes to be accepted by many decision makers with satisfaction. Most importantly, the linearity assumption in DEA may be problematic when the preference of decision maker cannot be represented as a linear function. Also we cannot find much research effort on the weight restriction methods that focus on the relation of each input and output (AR- II).

## **1.5 Organization of the dissertation**

This dissertation consists of 8 chapters. In chapter 3 and 4, we analyze characteristics of A-P model and cross-efficiency evaluation, which are frequently used when we don't have any prior relative weights of inputs and outputs. In chapter 5 and 6, we consider characteristics of DEA ranking models with cone-ratio and Wong and Beasley weight restrictions, both of which take decision maker's preferences into account. And in chapter 7,

we suggest alternative models for measuring overall efficiency. Finally, conclusions are provided in chapter 8.

The followings are the summary of contents of each chapter.

In chapter 2, we briefly introduce the CCR model with basic definitions, units invariance theorem and example as well as the weight restriction models suggested in previous DEA literature.

In chapter 3, to identify the characteristics of A-P model and cross-efficiency evaluation, we provide empirical ranking results in both models after describing their ranking criteria. Then we suggest specialization index (SI) that is computed using A-P multipliers and  $A_k$  score that is computed using cross-efficiency matrix to identify specialized DMUs. The result table used to find SI score clearly shows A-P model's characteristics. Also we examine the primary conclusions on the 1<sup>st</sup> ranker of cross-efficiency evaluation and show these conclusions are not always true.

Finally we propose using a biplot, which facilitates the comparison of characteristics of each model visually.

In chapter 4, we show that cross-efficiency evaluation in effect applies almost fixed weights in many of multiple-input, multiple-output cases, which is done as an extension of previous work [4] that focused on single-input, multiple-outputs case.

We develop an equation, which shows an efficiency score under fixed weighting scheme in multiple-input, multiple-output situation and analyze the difference between real cross-efficiency scores and those under fixed weighting both analytically and empirically.

In chapter 5, we define a property that shows the way to measure efficiency score when cone-ratio weight restrictions are applied under constant returns to scale with single- input, multiple-outputs (or multiple-inputs, single-output) in DEA. Based on this property we propose some graphical explanations of other DEA issues, 1) multiple solution problem 2) multipliers of cross-efficiency evaluation 3) target points under cone-ratio weight restrictions using one-input, two-output case in DEA.

In chapter 6, we analyze the characteristics of W/B weight restrictions theoretically and compare with those of C/R weight restriction empirically. We show that under W/B weight restriction, each DMU takes all different weight vectors and some DMUs may have limiting efficiency score.

In chapter 7, we present alternative models that can measure each of overall efficiency (OE) with cone-ratio weight restrictions and compare with previous models using examples. The only difference between the proposed and CCR model is the added cost (price) vector constraints, which results in the DEA models with cone-ratio weight restrictions.

## **CHAPTER 2. THE CCR AND WEIGHT RESTRICTION MODELS IN DEA**

### **2.1 Introduction**

In this chapter we will begin with explaining the CCR model and the several related important definitions and theorems. We included the brief explanation of the weight restrictions and its corresponding models in DEA in the last section of this chapter.

These can be found in many different sources. However, for clear and coherent explanations, much of the following framework is excerpted from recently published two textbooks by Cooper et al [11] and Thanassoulis [38].

This chapter is organized as follows. In section 2.2, the input-oriented CCR model and its dual form are presented. In section 2.3, the units invariance theorem is explained. In section 2.4, some fundamental definitions on the CCR model, i.e. production possibility set, CCR-efficiency (or technical efficiency), Pareto-Koopmans efficiency are presented. In section 2.5, the properties of input, output oriented CCR model and their relations under constant returns to scale are explained. In section 2.6, we suggest an example to explain most of above properties in CCR model. Finally in section 2.7, weight restriction models in DEA are suggested and explained.

### **2.2 The CCR Model**

The DEA method determines a measure of the relative efficiency of each DMU in comparison to all of the remaining DMUs under consideration. Given the data, we can measure the efficiency of each DMU one at a time and hence we need  $n$  optimizations.



That is for each DMU  $j_o$ , we formed the virtual input and output by yet unknown weights  $v_i, \mu_r$  [11]:

$$\text{Virtual input} = v_1 x_{1j_o} + \dots + v_m x_{mj_o}$$

$$\text{Virtual output} = \mu_1 y_{1j_o} + \dots + \mu_s y_{sj_o}$$

We then use linear programming to determine the weights that maximize the ratio  $\frac{\text{virtual output}}{\text{virtual input}}$ . That is, the objective of model (2.1: CCR-Fractional Program) is to obtain weights that maximize the ratio of DMU  $j_o$  the DMU being evaluated. The constraints prohibit the ratio of “virtual output / virtual input” from exceeding 1 for any DMU.

$$\begin{aligned}
 (\text{CCR-F}) \quad & \underset{u_r, v_i}{\text{Max}} \quad h_{j_o} = \frac{\sum_{r=1}^s \mu_r y_{rj_o}}{\sum_{i=1}^m v_i x_{ij_o}} \\
 \text{subject to} \quad & \frac{\sum_{r=1}^s \mu_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} \leq 1, \quad j = 1, \dots, n, \\
 & \mu_r, v_i \geq 0, \quad \forall r \text{ and } i
 \end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
 y_{rj} &= \text{amount of output } r \text{ from unit } j, & x_{ij} &= \text{amount of input } i \text{ to unit } j, \\
 \mu_r &= \text{the weight given to output } r, & v_i &= \text{the weight given to input } i, \\
 n &= \text{the number of units,} & s &= \text{the number of outputs,} \\
 m &= \text{the number of inputs}
 \end{aligned}$$

Using the fractional programming theory from Charnes and Cooper (Charnes-Cooper transformation), the above ratio-maximizing problem is equivalently transformed to the

following dual pair of linear programs (2.2 : CCR-Input Oriented) and (2.3 : Dual of 2.2)

$$\begin{aligned}
 \text{(CCR-I)} \quad & \text{Max}_{u_r} h_{j_0} = \sum_{r=1}^s \mu_r y_{rj_0} \\
 \text{subject to} \quad & \sum_{i=1}^m v_i x_{ij_0} = 1 \\
 & \sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, n \\
 & \mu_r, v_i \geq 0, \quad \forall r \text{ and } i
 \end{aligned} \tag{2.2}$$

$$\begin{aligned}
 \text{(CCR-ID)} \quad & \text{Min}_{\theta, \lambda} \theta_{j_0} \\
 \text{subject to} \quad & \sum_{j=1}^n \lambda_j x_{ij} - \theta_{j_0} x_{ij_0} \leq 0, \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} - y_{rj_0} \geq 0, \quad r = 1, \dots, s \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{2.3}$$

When we need to find out the slack value of each input / output variable, we have to solve model (2.4), which finds the solution that maximizes the sum of input excesses and output shortfalls while keeping  $\theta_{j_0} = \theta_{j_0}^*$ .

$$\begin{aligned}
 & \text{Max} \quad \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\
 \text{subject to} \quad & \sum_{j=1}^n \lambda_j x_{ij} - \theta_{j_0}^* x_{ij_0} + s_i^- = 0, \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} - y_{rj_0} - s_r^+ = 0, \quad r = 1, \dots, s \\
 & \lambda_j \geq 0, \quad s_i^- \geq 0, \quad s_r^+ \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{2.4}$$

In practice, solving model (2.3) and (2.4) sequentially is more preferable than solving model (2.2) directly because 1) The number of constraints in model (2.2) is  $n$  and that of

model (2.3) is  $m + s$ . Usually in DEA,  $n$  is much larger than  $m + s$  therefore from the computational aspect model (2.3) is preferable. 2) Sometimes we want to know the values of each input / output slack or  $\lambda$ , but from model (2.2) we cannot obtain those values. From the obtained  $\lambda$  value, we can see the reference set for each inefficient DMU, and finding the reference set has been considered as one of the most important advantages using DEA.

### 2.3 Units invariance theorem

In DEA, measured efficiencies are independent of the units of measurement used. This property can be stated as the following units invariance theorem [11].

“ The optimal values of  $\max h_{j_o} = h_{j_o}^*$  in (2.1) and (2.2) are independent of the units in which the inputs and outputs are measured provided those units are the same for every DMU ”.

In the single input and single output case, let's assume that (1) DMU A uses 3 units of input and produces 3 units of output thus show the ratio of (output/input) as 1 (technically efficient). (2) DMU B uses 5 units of input and produces 2 units of output thus show the ratio of (output/input) as  $2 / 5 = 0.4$  (technically inefficient).

If the unit of output is changed to 10 times of previous one, then the ratio of (output/input) as for DMU A will be  $30 / 3 = 10$  and that of DMU B will be  $20 / 5 = 4$ .

However the ratio won't change, i.e. the ratio of (output/input) of DMU B / the ratio of (output/input) of DMU A =  $4 / 10 = 0.4$ . Therefore the relative efficiency of each inefficient DMU is not affected by the choice of different unit of measure.

## 2.4 Some fundamental definitions on CCR model [11], [38]

### (1) Production possibility set

With the data sets in matrices  $\mathbf{X} = (\mathbf{x}_j)$  and  $\mathbf{Y} = (\mathbf{y}_j)$ , production possibility set  $P$  can be defined as follows satisfying (A-1) through (A-4).

$$P = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \geq \mathbf{X}\lambda, \mathbf{y} \leq \mathbf{Y}\lambda, \lambda \geq 0\} \quad (2.5)$$

(A-1) The observed activities  $(\mathbf{x}_j, \mathbf{y}_j)$  ( $j = 1, \dots, n$ ) belong to  $P$

(A-2) If an activity  $(\mathbf{x}, \mathbf{y})$  belongs to  $P$ , then the activity  $(t\mathbf{x}, t\mathbf{y})$  belongs to  $P$  for any positive scalar  $t$ . This property is called the constant returns-to-scale assumption.

(A-3) For an activity  $(\mathbf{x}, \mathbf{y})$  in  $P$ , any semi positive activity  $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$  with  $\bar{\mathbf{x}} \geq \mathbf{x}$  and  $\bar{\mathbf{y}} \leq \mathbf{y}$  is included in  $P$ .

(A-4) Any semi positive linear combination of activities in  $P$  belongs to  $P$ .

### (2) Definition of CCR-Efficiency, Technical Efficiency

These two definitions on CCR-Efficiency are equivalent [11]. Definition 1 applies when we use model (2.2) and definition 2 refers when we use model (2.3) and (2.4).

#### (Definition 1)

1) DMU  $j_o$  is CCR-efficient if  $h_{j_o}^* = 1$  and there exists at least one optimal

$$(\nu^*, \mu^*), \text{ with } \nu^* > 0 \text{ and } \mu^* > 0.$$

2) Otherwise, DMU  $j_o$  is CCR-inefficient.

#### (Definition 2)

- 1) If an optimal solution  $(\theta^*, \lambda^*, s^{-*}, s^{+*})$  of models (2.3) and (2.4) satisfies  $\theta^* = 1$  and zero-slack ( $s^{-*} = 0, s^{+*} = 0$ ), then DMU  $j_o$  is CCR-efficient
- 2) Otherwise, DMU  $j_o$  is CCR-inefficient.

### (3) Pareto-Koopmans Efficiency

- 1) A DMU is fully efficient if and only if it is not possible to improve any input or output without worsening some other input or output.
- 2) If DMU  $j_o$  is CCR-efficient, then it is Pareto-Koopmans efficient.

## 2.5 Input Oriented / Output Oriented models

There are two kinds of different measures of efficiency in DEA. Depending on whether inputs or outputs are more controllable, different measure of efficiency are appropriate. One is the efficiency by input orientation and the other is that by output orientation.

If inputs are more controllable, input orientation is appropriate, and if outputs are more controllable, output orientation is more appropriate. Model (2.2) ~ (2.4) corresponds to input orientation and the following model (2.6) ~ (2.7) corresponds to the output orientation [11], [38]. Under the constant returns to scale, both input and output oriented models produce the same efficiency score. However, under variable returns to scale, these two orientation may produce different efficiency scores.

$$\begin{aligned}
 \text{(CCR-O)} \quad & \underset{v_i}{\text{Min}} \quad h_{j_o} = \sum_{i=1}^m v_i x_{ij_o} \\
 \text{subject to} \quad & \sum_{r=1}^s \mu_r y_{rj} = 1
 \end{aligned} \tag{2.6}$$

$$\sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, n$$

$$\mu_r, v_i \geq 0, \quad \forall r \text{ and } i$$

(CCR-OD) 
$$\begin{aligned} & \text{Max}_{\theta, \lambda} \theta_{j_0} \\ \text{subject to} & \quad x_{ij_0} - \sum_{j=1}^n \lambda_j x_{ij} \geq 0, \quad i = 1, \dots, m \\ & \quad \theta_{j_0} y_{rj_0} - \sum_{j=1}^n \lambda_j y_{rj} \leq 0, \quad r = 1, \dots, s \\ & \quad \lambda_j \geq 0, \quad j = 1, \dots, n \end{aligned} \quad (2.7)$$

That is, 1) The technical input efficiency measure  $h_{j_0}^{*(I)}$  obtained from model (2.2) and  $\theta_{j_0}^{*(I)}$  obtained from model (2.3) are equal 2) The technical output efficiency measure  $1/h_{j_0}^{*(O)}$  obtained from model (2.6) and  $1/\theta_{j_0}^{*(O)}$  obtained from model (2.7) are equal 3) The efficiency score obtained from above two orientation models are equal under the constant return to scale technology, i.e.  $h_{j_0}^{*(I)} = 1/h_{j_0}^{*(O)}$ ,  $\theta_{j_0}^{*(I)} = 1/\theta_{j_0}^{*(O)}$ .

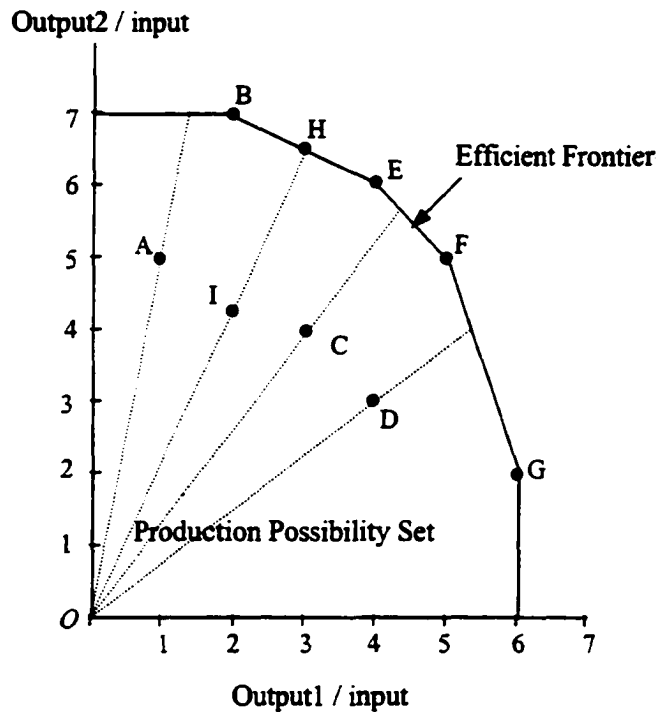
## 2.6 Example

Table 2.1 shows the data that is from Cooper et al 's text [11] with adding 2 more DMUs. The input is the number of employees and the two outputs are the number of customers and the amount of sales at 9 branch offices.

**Table 2.1 Data of example 2.1**

Store		A	B	C	D	E	F	G	H	I
Employees	$x$	1	1	1	1	1	1	1	1	1
Customers	$y_1$	1	2	3	4	4	5	6	3	2
Sales	$y_2$	5	7	4	3	6	5	2	6.5	4.333

The number of customers per salesman has the unit =10, and sales per salesman has the unit = \$100,000. Figure 2.1 shows the production possibility set and the efficient frontier composed of DMUs B, H, E, F, and G.



**Figure 2.1 One input and two outputs case**

Table 2.2 shows the results of example 2.1. The second column shows the CCR-efficiency score and the third column shows the reference set of each DMUs, which can be obtained by solving dual form (2.3). The reference set of each inefficient DMU can be obtained from model (2.3) : i.e.

for DMU A :  $\lambda_B = 0.714$ , for DMU C :  $\lambda_E = 0.5$ ,  $\lambda_F = 0.2$ ,

for DMU D :  $\lambda_F = 0.5$ ,  $\lambda_G = 0.25$ , and for DMU I :  $\lambda_B = 0.333$ ,  $\lambda_E = 0.333$ .

And the slack value calculated from model (2.4) is for DMU A :  $s_{A,1}^- = 0.4286$ .

The CCR multipliers columns are showed in the following 3 cases.

- CCR multipliers (I) : the case when example 2.1 data is used as they are.
- CCR multipliers (II) : the case when the unit of (output 1 / input ) is changed from \$100,000 to \$10,000 (i.e. the value is changed to 10 times to previous one), then each DMU's multiplier value of  $\mu_1$  is also changed to 1/10 times of those of case (I). But the efficiency score is the same as case (I) from units invariance theorem.
- CCR multipliers (III) : the case when normalizing constant is replaced from 1 to 10. In this case the multiplier value of  $\mu_1$  and  $\mu_2$  , and the efficiency scores are all changed to 10 times of case (I). But final efficiency score is the same as before because for DMU A : efficiency score =  $7.14 / 10 = 0.714$ .

**Table 2.2 Results of example 2.1**

DMU	CCR	Ref. Set	CCR Multipliers (I)			CCR Multipliers (II)			CCR Multipliers (III)		
			$v$	$\mu_1$	$\mu_2$	$v$	$\mu_1$	$\mu_2$	$v$	$\mu_1$	$\mu_2$
A	0.714	B	1.0	0	0.1429	1.0	0	0.1429	10	0	1.429
B	1	B	1.0	0	0.1429	1.0	0	0.1429	10	0	1.429
C	0.700	E, F	1.0	0.1	0.1	1.0	0.01	0.1	10	1	1
D	0.750	F, G	1.0	0.15	0.05	1.0	0.015	0.05	10	1.5	0.5
E	1	E	1.0	0.1	0.1	1.0	0.01	0.1	10	1	1
F	1	F	1.0	0.1	0.1	1.0	0.01	0.1	10	1	1
G	1	G	1.0	0.15	0.05	1.0	0.015	0.05	10	1.5	0.5
H	1	H	1.0	0.0625	0.125	1.0	0.00625	0.125	10	0.625	1.25
I	0.667	B, E	1.0	0.0625	0.125	1.0	0.00625	0.125	10	0.625	1.25

It is interesting to note that DMUs C, E and F have the same multiplier weights for  $\mu_1$  and  $\mu_2$ . DMUs A, B, H and I have larger weights on  $\mu_2$ . DMUs D and G have larger weights on  $\mu_1$ . From Figure 2-1, we can confirm above weighting scheme in DEA.



## 2.7 Weight restriction models in DEA

Using the CCR model has the practical advantage that the user need not identify prior relative values of inputs and outputs. Unfortunately, the imputed input / output values in this manner may be problematic when the user has certain value judgments which should be taken into account in the assessment and those values are not accord with the imputed values of CCR model. Thanassoulis [38] mentioned some of the circumstances when we would wish to incorporate value judgment in a DEA assessment as follows.

- 1) Imputed values may not accord well with prior views on the marginal rates of substitution and / or transformation of the factors of production.
- 2) Certain inputs and outputs may have a special interdependence within the production process modeled.
- 3) We may wish to arrive at some notion of 'overall efficiency'.
- 4) We may wish to discriminate between Pareto-efficient units. The restrictions on relative worth of inputs and outputs can help to discriminate between Pareto-efficient DMUs.

The comprehensive range of weight restrictions that can be used to incorporate value judgments in DEA under constant returns to scale are summarized in model (2.8) [38].

(1) AR-I (Assurance Regions type I) :  $r_1 \sim r_4$

- Each restriction links either only input, or alternatively only output weights.
- Use of form  $r_2$  and  $r_4$  is more prevalent in practice, reflecting valid marginal rates of substitution as perceived by the decision maker.
- The name Assurance Regions type I, type II are due to Thomson et al [39].

(2) AR-II (Assurance Region type II) :  $r_5$ 

- This type shows the relationships between input and output weights

$$(CCR-WR) \quad \text{Max}_{u_r} h_{j_0} = \sum_{r=1}^s \mu_r y_{rj_0}$$

$$\text{subject to} \quad \sum_{i=1}^m v_i x_{ij_0} = 1 \quad (2.8)$$

$$\sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, n$$

$$\left. \begin{aligned} \kappa_i v_i + \kappa_{i+1} v_{i+1} &\leq v_{i+2} && : r1 \\ \alpha_i \leq \frac{v_i}{v_{i+1}} &\leq \beta_i && : r2 \\ \varphi_r \mu_r + \varphi_{r+1} \mu_{r+1} &\leq \mu_{r+2} && : r3 \\ \theta_r \leq \frac{\mu_r}{\mu_{r+1}} &\leq \gamma_r && : r4 \end{aligned} \right\} \quad (\text{AR- I}) : r_1 \sim r_4$$

$$\delta_i v_i \geq \mu_r \quad : r5 \quad (\text{AR- II}) : r_5$$

$$\left. \begin{aligned} \omega_i \leq v_i &\leq \tau_i && : r6 \\ \rho_r \leq \mu_r &\leq \eta_r && : r7 \end{aligned} \right\} \quad \text{Absolute weight restrictions}$$

$$\mu_r, v_i \geq 0, \quad \forall r \text{ and } i$$

(3) Absolute weights restrictions :  $r_6 \sim r_7$ 

- This type is introduced to prevent inputs or outputs from being over emphasized or ignored in the analysis

- However these absolute bounds are dependent on the normalization constant, and may not maximize the relative efficiency of assessed DMU. The side effects of this absolute weight restriction are recently proposed in detail by Podinovski [23].

## **CHAPTER 3. A-P MODEL AND CROSS-EFFICIENCY EVALUATION**

### **3.1 Introduction**

As we described in chapter 1, unless we have weight restrictions on input or output variables obtained by human value judgments, DEA ranking methods must find ranking criteria that can restrict or adjust the weights appropriately to make a full ranking. In fact, many DEA ranking models start from the assumption that there are no given preferences (criteria), which are often the cases in real life applications, and therefore the ranking criteria of each model vary by the method each model used.

Among the DEA ranking models that try to make a ranking without given weight restriction, Andersen - Petersen model (A-P model) [3] and the cross-evaluation model [13] have been the most frequently used models in the DEA literature. However each model often makes far different ranking results in many applications. This motivates research on the characteristics of each model, which hopefully can help decision makers to make a better decision.

On characteristics of the 1<sup>st</sup> ranker of each model, several points emerge from previous DEA literature. First, the 1<sup>st</sup> ranker in A-P may be a specialized DMU. This has been considered as one of the important problematic areas of A-P model. However, no way to identify specialized DMU has been suggested except deciding by A-P multipliers. Second, the 1<sup>st</sup> ranker in cross-efficiency evaluation is considered as a “winner with many competitors”, the least maverick in the sample (i.e. all-round performer). Doyle et al [13], [14] suggested these characteristics, which supported this model to be used as a popular tool in many DEA applications. To show characteristics of each model, in this chapter we provide

empirical ranking results in both models after describe their ranking criteria. Then we suggest specialization index (SI) in A-P model and  $A_k$  score in cross-efficiency evaluation to identify specialized DMU. Also we examine the above conclusions on the 1<sup>st</sup> ranker of cross-efficiency evaluation and show these primary conclusions are not always true. Finally we suggest using a biplot, which facilitates the comparison of characteristics of each model visually. Based on the fact that cross-efficiency evaluation uses almost fixed weights in many of multiple-input, multiple-output cases, we can represent the weight direction of cross-efficiency evaluation in the biplot.

The rest of this chapter is organized as follows. In section 3.2, two sets of application data are introduced. In section 3.3, each model's ranking criteria, performing procedure and the important characteristics shown in previous literature are presented.

In section 3.4, we suggest specialization index (SI), which enables the identification of specialized DMUs in A-P model. The result table used to find SI clearly shows A-P model's characteristics. Also in section 3.5, the primal claims on the characteristics of 1<sup>st</sup> ranker in cross-efficiency evaluation are examined and the  $A_k$  score is suggested as a replacement of Maverick index. Empirical results are used to compare the 1<sup>st</sup> ranker in cross-evaluation with that under restriction of equal input, output weights along with the explanation of simple case in which cross-efficiency makes unexpected ranking result.

In section 3.6, we describe how to develop biplots. Finally, conclusions are provided in section 3.7.

### 3.2 Application examples

In this dissertation, we will mostly use the following two application examples (1) and (2) to illustrate the results, 1) Performance evaluation of FMS (Flexible Manufacturing System) in [29] 2) Car selection problem in [20], [24]. Some other data found in [12], [21], [26], [42] are also used to verify the results in each chapter. The data of FMS [29] and Car selection problem [20], [24] are shown in Table A.1 in Appendix.

(1) Performance evaluation of FMS (Two inputs, four outputs, 12 DMUs)

- Input 1 includes the annual operating and depreciation costs, which are measured in units of one hundred thousand dollars. Input 2 is the floor space requirements of each specific system, which are measured in thousands of square feet.
- Output 1 is the qualitative benefits, which is measured as a percentage, Output 2 is the WIP which is measured in units of 10, Output 3 is the average number of tardy jobs, which is measured in percentages, Output 4 is the average yield, which are measured in units of 100

They demonstrated that all the output measures could be derived by computing the performance of the respective system against that of the existing system. Therefore the output 2 and 3, output improvements are measured by the amount that can be reduced by the respective system.

They also stated that the output measures could be obtained by AHP (Analytic Hierarchy Process) or simulation study, and the input measures can be obtained by company's accounting process. But in this paper, we will limit our attention to the characteristics each DEA model and thus more detailed explanation of input or output variable selection process in FMS is out of the scope of this dissertation.

## (2) Car selection problem (4 inputs, 2 outputs, 28 DMUs)

The data of car selection problem was introduced in [20], [24] to show geometrical representation method of multi-criteria decision problems. The 6 criteria used in that problem were F1 : Price (1000FF) (minimize), F2 : DIN (Deutsches Institut für Normung ; German Institute for standardization) power (maximize), F3 : Fiscal power (minimize), F4 : Maximum speed (maximize), F5: Urban fuel consumption (minimize), F6 : 90km/h fuel consumption (minimize). Mareschal [20] selected the top 10 best cars (6, 3, 12, 8, 14, 5, 1, 2, 17, 15) using one of the MCDM methods (PROMETHEE II). In this dissertation, we changed all of minimizing criteria as inputs and maximizing criteria as outputs to apply to DEA models.

## (3) Other application examples

The other application data found in previous DEA literature we use in this dissertation are as follows. 1) Economic performance of Chinese cities data (2 inputs, 3 outputs, 18 DMUs) [42], 2) Location of hydro electrical power station data (4 inputs, 2 outputs, 6 DMUs) [12], 3) Location of solid waste management system data (5 inputs, 3 outputs, 22 DMUs) [26], 4) Evaluating regions in Serbia data (4 inputs, 4 outputs, 30 DMUs) [21]. In this dissertation, we limit our focus to analyze the characteristics of DEA ranking models. All application data are shown in Table A.1 in the appendix but the detailed explanation of each data is out of scope of this study.

### 3.3 Review on the characteristics of each model

#### 3.3.1 A-P model

##### (1) Ranking criteria

Andersen and Petersen (1993) [3] suggested a model that can discriminate between all DMUs without requiring a priori weights on inputs and outputs. The ranking criteria of the model can be summarized as follows.

- 1) Compare the unit under evaluation with a linear combination of all other units in the sample, i.e. the DMU itself is excluded
- 2) The score reflects the radial distance from the DMU under evaluation to the production frontier estimated with that DMU excluded from the sample.

Therefore the A-P efficiency score can be interpreted as the maximum proportional increase in inputs preserving efficiency (or maximum proportional decrease in outputs preserving efficiency).

In Fig 3.1, A-P efficiency score of DMU C is measured by  $\frac{OC'}{OC}$ , and those of DMU B and D are  $\frac{OB'}{OB}$  and  $\frac{OD'}{OD}$  respectively. Therefore all of the CCR-efficient DMU's A-P efficiency scores are greater than or equal to 1. On the other hand, the A-P efficiency scores of CCR-inefficient DMUs are the same as their CCR efficiency score.

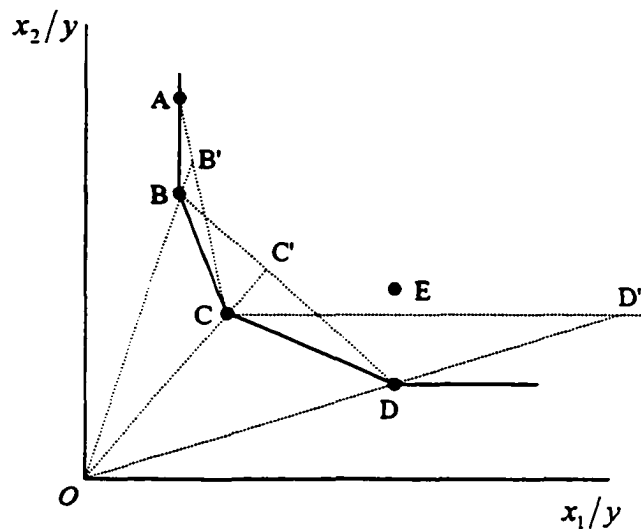


Figure 3.1 A-P model

## (2) Performing procedure

The A-P efficiency score explained above can be simply calculated by either of the following LPs (3.1) or (3.2). Program (3.1) represents the A-P model with input orientation and model (3.2) is the dual form of equation (3.1). The only difference with CCR-I model is the omission of the constraint, which belongs to the DMU being evaluated.

$$\begin{aligned}
 \text{(AP-I)} \quad & \text{Max}_{\mu_r} h_{j_0} = \sum_{r=1}^s \mu_r y_{rj_0} \\
 \text{subject to} \quad & \sum_{i=1}^m v_i x_{ij_0} = 1 \\
 & \sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, n, \quad j \neq j_0 \\
 & \mu_r, v_i \geq 0, \quad \forall r \text{ and } i
 \end{aligned} \tag{3.1}$$

$$\begin{aligned}
 \text{(AP-ID)} \quad & \text{Min}_{\theta, \lambda} \theta_{j_0} \\
 \text{subject to} \quad & \sum_{\substack{j=1 \\ j \neq j_0}}^n \lambda_j x_{ij} - \theta_{j_0} x_{ij_0} \leq 0, \quad i = 1, \dots, m \\
 & \sum_{\substack{j=1 \\ j \neq j_0}}^n \lambda_j y_{rj} - y_{rj_0} \geq 0, \quad r = 1, \dots, s \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{3.2}$$

(3) The characteristics of 1<sup>st</sup> ranker from A-P model

About the A-P model, three problematic areas have been recognized by previous researches [1], [3], [37], [40]. However despite these drawbacks, possibly because of the simplicity of the concept, many published papers in DEA research have used this approach to make a ranking of sample DMUs [1].



The disclosed problematic areas are

- 1) Since A-P takes the objective function value directly as a rank score of each DMU, each DMU is evaluated according to different weights [1].
- 2) The A-P model can give “specialized” DMUs an excessively high ranking [1],[3],[37].
- 3) Under certain conditions, the A-P model can yield an infeasible solutions [40].

Among the above three problems, the second problem has been most frequently indicated. The only difference between A-P and CCR model is just the omission of DMU that is being evaluated in the constraints. Therefore A-P model is very similar to the CCR model in weighting scheme that gives more emphasis on the flexibility in choosing weights to show technical efficiency. All of the CCR-inefficient DMUs have the same results on weights and efficiency score by the A-P model as those by the CCR model.

However for the CCR-efficient DMUs, the A-P efficiency scores are decided by the score of second follower with respect to each weight vector. Therefore a certain DMU, which has a unique feature in its weighting scheme, can have a high score by A-P model. This is generally regarded as a problem of A-P model [1], [3], [37] such that A-P model can give “specialized” DMUs an excessively high ranking.

Although the second problem was frequently mentioned in previous research, the way for identifying the specialized DMU has not been clearly identified. In much DEA literature, the possibility of specialized DMU has been explained by A-P multipliers that when a small number of variable weights have high values and all the others have 0 (or almost 0) weights, that DMU can be a specialized DMU. However in many cases, to decide a specialized DMU only by the assigned weights is not so clear.

Table 3.1 shows the result of A-P efficiency scores and its multipliers on the FMS data.

Among the CCR efficient DMUs (1, 2, 4, 5, 6, 7 and 9), DMU 9 seems to be the most specialized since it has only two positive multipliers while others have at least three positive multipliers. However just from the multipliers in Table 3.1, we cannot make sure that which DMU is a specialized DMU since many DMUs also have small number of positive multipliers.

**Table 3.1 A-P multipliers of FMS data**

(FMS)	Efficiency	Input weights		Output weights			
	(A-P)	$v_1$	$v_2$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$
1	1.045	0.0549	0.0132	0.0204	0.0042	0	0
2	1.093	0	0.2222	0.0026	0	0	0.0332
3	0.982	0.0850	0	0.0283	0	0.0162	0.0009
4	1.134	0.0491	0.1208	0	0	0	0.0454
5	1.159	0.0757	0.0738	0	0	0.0966	0
6	1.028	0.2088	0	0	0	0.0671	0.0420
7	1.060	0.1610	0	0.0509	0.0120	0.0042	0
8	0.961	0.0758	0.0262	0.0211	0.0040		0.0118
9	1.432	0.2725	0	0	0	0	0.0791
10	0.953	0.0877	0.0310	0	0.0278	0	0
11	0.983	0.0564	0	0.0206	0	0	0.0031
12	0.801	0.0593	0.0191	0.0117	0	0.0049	0.0111
mean		0.0989	0.0422	0.0130	0.0040	0.0172	0.0189

### 3.3.2 Cross-efficiency evaluation

#### (1) Ranking criteria

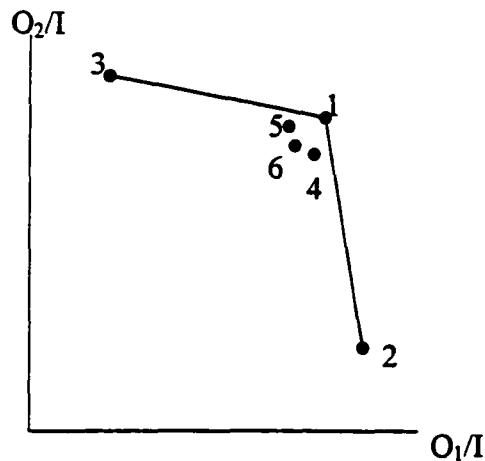
Doyle and Green suggested their cross-efficiency formulations (aggressive and benevolent formulations) [13] in 1994 as an extension of [30] in 1986 and showed that aggressive and benevolent formulations are highly correlated.

In [13], they reviewed the favorable aspects of using cross-efficiency such that 1) each DMU is rated not only by its weighting scheme but also other DMU's weighting scheme,

making it far more difficult to have ties and more likely to create a unique ordering 2) it can overcome the problem of selecting maverick DMU with suggesting a maverick index 3) Maverick index identifies the all-round best performer.

In their subsequent paper [14], they signify the essence of the ranking criteria of cross-efficiency as follows.

“ To get a first rank in cross-efficiency evaluation is equivalent to winning in a big race with many competitors. It could be said that coming second in a race where there were a thousand competitors must surely be better than coming first in a walkover”.



**Figure 3.2 The criteria of cross-efficiency evaluation**

Figure 3.2 represents their concept of cross-efficiency evaluation based on the example 3.1 data (shown in Table 3.3), i.e. it considers DMU 1 as more ‘right stuff’ than DMU 2 or 3. Because DMU 1 is referred to be more appropriate stuff with respect to the basic two cross-efficiency criteria, (1) being efficient and also (2) being core. Moreover in the cross-efficiency DMUs 4, 5 and 6 might be superior to DMU 2 and 3, because, in this example, the second criterion dominates the first criterion.

## (2) Performing procedure

To get the cross-efficiency score we need to perform three step calculations.

*Step 1.* Find the CCR efficiency score  $\theta_{j_o}$  for DMU  $j_o$  from equation (3.3)

$$\begin{aligned}
 \text{(CCR-I)} \quad & \underset{u_r}{\text{Max}} \theta_{j_o} = \sum_{r=1}^s \mu_{r,j} y_{r,j_o} \\
 & \text{subject to} \\
 & \sum_{i=1}^m v_{i,j} x_{ij_o} = 1 \\
 & \sum_{r=1}^s \mu_{r,j} y_{r,j} - \sum_{i=1}^m v_{i,j} x_{i,j} \leq 0, \quad j = 1, \dots, n \\
 & \mu_{r,j}, v_{i,j} \geq 0, \quad \forall r \text{ and } i
 \end{aligned} \tag{3.3}$$

*Step 2.* Find the virtual multipliers with restriction that the efficiency score for DMU  $j_o$  is set to the CCR efficiency score  $\theta_{j_o}$  from equation (3.4)

$$\begin{aligned}
 \text{(CEM-I)} \quad & \underset{u_r}{\text{Min}} \sum_{r=1}^s \left( \mu_{r,j} \sum_{j \neq j_o} y_{r,j} \right) \\
 & \text{subject to} \\
 & \sum_{i=1}^m \left( v_{i,j} \sum_{j \neq j_o} x_{i,j} \right) = 1, \\
 & \sum_{r=1}^s \mu_{r,j} y_{r,j} - \sum_{i=1}^m v_{i,j} x_{i,j} \leq 0, \quad \forall j \neq j_o \\
 & \sum_{r=1}^s \mu_{r,j} y_{j_o} - \theta_{j_o} \sum_{i=1}^m v_{i,j} x_{i,j_o} = 0, \\
 & \mu_{r,j}, v_{i,j} \geq 0, \quad \forall r \text{ and } i
 \end{aligned} \tag{3.4}$$

*Step 3.* Find the cross-efficiency score ( $CE_k$ ) of DMU  $k$  by equation (3.5) based on the element ( $E_{j,k}$ ) of cross-efficiency matrix.

$$E_{j,k} = \frac{\sum_{r=1}^s \mu_{r,j} y_{r,k}}{\sum_{i=1}^m v_{i,j} x_{i,k}}, \quad CE_k = \frac{1}{n} \sum_{j=1}^n E_{j,k}, \quad A_k = \frac{1}{n} \sum_{j=1}^n E_{k,j} \quad (3.5)$$

Table 3.2 shows an example of cross-efficiency matrix for 6 DMUs.  $CE_k$  is the cross efficiency score and can be said that averaged appraisal by peers (peer appraisal), and  $A_k$  can be said the averaged appraisal of peers.  $E_{j,k}$  is the DMU  $k$ 's efficiency score when DMU  $j$ 's multipliers are used. Table 3.3 shows example 3.1 data and cross efficiency scores of each DMU.

**Table 3.2 Example of cross-efficiency matrix for 6 DMUs**

DMU	1	2	3	4	5	6	$A_k$
1	$E_{11}$	$E_{12}$	$E_{13}$	$E_{14}$	$E_{15}$	$E_{16}$	$A_1$
2	$E_{21}$	$E_{22}$	$E_{23}$	$E_{24}$	$E_{25}$	$E_{26}$	$A_2$
3	$E_{31}$	$E_{32}$	$E_{33}$	$E_{34}$	$E_{35}$	$E_{36}$	$A_3$
4	$E_{41}$	$E_{42}$	$E_{43}$	$E_{44}$	$E_{45}$	$E_{46}$	$A_4$
5	$E_{51}$	$E_{52}$	$E_{53}$	$E_{54}$	$E_{55}$	$E_{56}$	$A_5$
6	$E_{61}$	$E_{62}$	$E_{63}$	$E_{64}$	$E_{65}$	$E_{66}$	$A_6$
$CE_k$	$CE_1$	$CE_2$	$CE_3$	$CE_4$	$CE_5$	$CE_6$	

The performing procedure of cross-efficiency evaluation is as follows.

1) Finding multipliers of each DMU

For example, when we calculate the multipliers of DMU 1, we solve the following LP

$$\text{Min} \quad (11.6 + 2.8 + 10.5 + 10.1 + 10.2) \mu_1 + (2.5 + 12.8 + 11.6 + 11.8 + 11.5) \mu_2$$

$$\text{subject to} \quad 5\nu = 1$$

$$11.6 \mu_1 + 2.5 \mu_2 - \nu \leq 0$$

$$2.8 \mu_1 + 12.8 \mu_2 - \nu \leq 0$$

$$\vdots$$

$$10.2 \mu_1 + 11.5 \mu_2 - \nu \leq 0$$

$$10.7 \mu_1 + 12.0 \mu_2 - 1.00 \nu = 0$$

$$\nu, \mu_1, \mu_2 \geq 0$$

The answer is  $\nu = 0.2$ ,  $\mu_1 = 0.001548$ ,  $\mu_2 = 0.015286$

These are multipliers that can minimize the sum of all the other DMU's virtual output subject to the sum of all the other DMU's virtual input is 1.

2) Make the cross-efficiency matrix and find the cross-efficiency score

Using these multipliers of each DMU, we can make the cross-efficiency matrix.

For example, DMU 1's cross-efficiency score will be calculated as follows.

**Table 3.3 Example 3.1 data and cross efficiency scores**

Data				CEM Multiplier - Input Oriented				
DMU	$x$	$y_1$	$y_2$	DMU	CCR	$\nu_1$	$\mu_1$	$\mu_2$
1	1	10.7	12.0	1	1	0.2	0.0015	0.0153
2	1	11.6	2.5	2	1	0.2	0.0172	0
3	1	2.8	12.8	3	1	0.2	0	0.0156
4	1	10.5	11.6	4	0.9799	0.2	0.0169	0.0016
5	1	10.1	11.8	5	0.9800	0.2	0.0015	0.0153
6	1	10.2	11.5	6	0.9579	0.2	0.0015	0.0153
				mean		0.2	0.0065	0.0105

DMU	1	2	3	4	5	6
1	1.000	0.281	1.000	0.968	0.980	0.958
2	0.922	1.000	0.241	0.905	0.871	0.879
3	0.938	0.195	1.000	0.906	0.922	0.898
4	1.000	1.000	0.339	0.980	0.948	0.954
5	1.000	0.281	1.000	0.968	0.980	0.958
6	1.000	0.281	1.000	0.968	0.980	0.958
mean	0.977	0.506	0.763	0.949	0.947	0.934

$$CE_1 = \frac{1}{6} \times \left[ \frac{(10.7 \times 0.0015) + (12 \times 0.0153)}{1 \times 0.2} + \frac{(10.7 \times 0.0172) + (12 \times 0)}{1 \times 0.2} + \dots \right. \\ \left. \dots + \frac{(10.7 \times 0.0015) + (12 \times 0.0153)}{1 \times 0.2} \right]$$

$$\begin{aligned}
&= \frac{1}{6} \times [ \text{(Efficiency score using DMU 1's multipliers)} + \text{(Efficiency score using} \\
&\quad \text{DMU 2's multipliers)} + \dots + \text{(Efficiency score using DMU 6's} \\
&\quad \text{multipliers)} ] \\
&= \frac{1}{6} (1.00 + 0.922 + 0.938 + 1.00 + 1.00 + 1.00) = 0.977
\end{aligned}$$

In the cross-efficiency matrix, the diagonal element should represent (equal) to the corresponding CCR efficiency score (self appraisal) of each DMU.

(3) The characteristics of 1<sup>st</sup> ranker from cross-efficiency evaluation

Until now, cross-efficiency evaluation has been used in many DEA applications [21], [26], [29] and it became one of the most popular tools in DEA research. Despite these research efforts, the main idea (or the characteristics of 1<sup>st</sup> ranker) on cross-efficiency evaluation such that it selects the all-round performer as 1<sup>st</sup> ranker, which is the least maverick in the sample, has not been doubted except a recent research [4].

In [4], Anderson et al demonstrated a negative aspect of cross-efficiency such that “cross-efficiency in effect applies a fixed set of weights to all DMUs in single-input, multiple outputs situation and this may be unrealistic”.

Since in [13], they didn't rigorously define the ‘all-round performer, which is also used in other research as ‘overall performer’ possible ways to understand this term include 1) all-round performer is the best performer which obtain the highest average score by peer appraisal ( $CE_k$ ) in the suggested cross-efficiency matrix or 2) it is also the winner with the most competitors as shown in Figure 3.2.

However, they do rigorously define the Maverick index (3.6) that can identify the all-round performers.

$$M_k = (E_{kk} - e_k) / e_k, \quad e_k = 1 / (n - 1) \sum_{s \neq k} E_{sk} \quad (3.6)$$

The order in Maverick index doesn't exactly correspond with that of cross-efficiency score in all DMUs, but in the case of CCR efficient DMUs the order is proven to be the same. That is, the higher cross-efficiency score of DMU  $j$  shows the smaller Maverick index it has. This implies that the 1<sup>st</sup> ranker in cross-evaluation is the least maverick and is the most all-round performer.

### 3.4 Identification of specialized performer

Since the ranking results between A-P model and cross-efficiency evaluation are often quite different and sometimes make the same 1<sup>st</sup> ranker, we need to identify the characteristics of each ranking model for decision makers to make a better decision.

To compare the ranking results from both models, we use the Spearman coefficient of rank correlations. Equation (3.7) represents the Spearman coefficient of rank correlations and Table 3.4 reports the nonparametric statistical test of the relationship between rankings under A-P model and cross-efficiency evaluation.

$$r_s = 1 - \frac{6 \sum_{i=1}^n d_i^2}{n(n^2 - 1)} \quad (3.7)$$

where,  $n$  = number of observations,  $d_i = S_i - R_i$  ( $S_i$  is the rank of A-P,  $R_i$  is the rank of cross-efficiency evaluation)

The 6 application data in previous DEA literature are used to compare the ranking results. That is, 1) FMS selection [29], 2) Car selection [20],[24], 3) Location of hydro electrical power station [12], 4) Location of solid waste management system [26],



5) Economic performance of Chinese cities [42], 6) Evaluating regions in Serbia [21].

Among these 6 data, two results (3 and 6) are appeared as rather high correlation  $r_s \geq 0.88$  but all the other results are appeared rather low correlations.

**Table 3.4 Spearman rank statistic in 6 application examples**

Applications	1	2	3	4	5	6
$r_s$	0.461	0.434	0.984	-0.005	0.356	0.884

Also when we compare the upper 3 rankers in 6 examples, we can see that these two models often make far different ranking results. Table 3.5 shows the results of upper 3 rankers, where  $n$  = number of DMUs, the parenthesized number under the column of rank\* represents the rank of other model.

**Table 3.5 Comparison of ranking in 6 application examples**

Application 1 (n=12)				Application 2 (n=28)				Application 3 (n=6)				
		DMU	rank*			DMU	rank*			DMU	rank*	
<b>A-P</b>	1 <sup>st</sup>	9	(12)	1 <sup>st</sup>	1	(13)	1 <sup>st</sup>	2	(6)	1 <sup>st</sup>	2	(6)
	2 <sup>nd</sup>	5	(1)	2 <sup>nd</sup>	21	(21)	2 <sup>nd</sup>	5	(1)	2 <sup>nd</sup>	5	(1)
	3 <sup>rd</sup>	4	(3)	3 <sup>rd</sup>	25	(23)	3 <sup>rd</sup>	4	(3)	3 <sup>rd</sup>	4	(3)
<b>CE</b>	1 <sup>st</sup>	5	(2)	1 <sup>st</sup>	6	(14)	1 <sup>st</sup>	5	(2)	1 <sup>st</sup>	5	(2)
	2 <sup>nd</sup>	1	(6)	2 <sup>nd</sup>	12	(11)	2 <sup>nd</sup>	6	(5)	2 <sup>nd</sup>	6	(5)
	3 <sup>rd</sup>	4	(3)	3 <sup>rd</sup>	3	(10)	3 <sup>rd</sup>	4	(3)	3 <sup>rd</sup>	4	(3)
Application 4 (n=22)				Application 5 (n=18)				Application 6 (n=30)				
		DMU	rank*			DMU	rank			DMU	rank*	
<b>A-P</b>	1 <sup>st</sup>	9	(1)	1 <sup>st</sup>	2	(1)	1 <sup>st</sup>	30	(6)	1 <sup>st</sup>	30	(6)
	2 <sup>nd</sup>	3	(20)	2 <sup>nd</sup>	10	(3)	2 <sup>nd</sup>	9	(13)	2 <sup>nd</sup>	9	(13)
	3 <sup>rd</sup>	21	(15)	3 <sup>rd</sup>	6	(2)	3 <sup>rd</sup>	20	(3)	3 <sup>rd</sup>	20	(3)
<b>CE</b>	1 <sup>st</sup>	9	(1)	1 <sup>st</sup>	2	(1)	1 <sup>st</sup>	23	(4)	1 <sup>st</sup>	23	(4)
	2 <sup>nd</sup>	6	(20)	2 <sup>nd</sup>	6	(3)	2 <sup>nd</sup>	24	(7)	2 <sup>nd</sup>	24	(7)
	3 <sup>rd</sup>	8	(9)	3 <sup>rd</sup>	10	(2)	3 <sup>rd</sup>	20	(3)	3 <sup>rd</sup>	20	(3)

For example, in the FMS example, the 1<sup>st</sup> ranker in A-P model (DMU 9) is the 12<sup>th</sup> ranker in cross-efficiency evaluation. Also, the 1<sup>st</sup> ranker in cross-efficiency evaluation (DMU 5) is the 2<sup>nd</sup> ranker in A-P model. It is interesting to see that while these two models often make far different ranking results, which is shown in bold number in Table 3.5, they sometimes make the same 1<sup>st</sup> ranker (Application 4 and 5) or similar ranking (Application 4).

In previous section, we indicated the problematic area of the A-P model such that A-P model can give “specialized” DMUs an excessively high ranking. In this section, we suggest specialization index (SI), which enables the identification of specialized DMUs.

Specialization index ( $SI_k$ : averaged appraisal of peers) can be defined similar to the way of finding ( $A_k$ : averaged appraisal of peers) in a cross-efficiency matrix. However to calculate  $SI_k$ , we have to use cone-ratio constraints of A-P multipliers to find each element  $E_{j,k}$  in the matrix. When we use A-P multipliers directly, it may result in an infeasible solution for CCR efficient DMUs.

The calculating procedure for finding  $SI_k$  can be summarized as follows.

In step 1, find each DMU's A-P multipliers, in step 2, calculate each DMU's efficiency score  $E_{j,k}$  using DMU  $k$ 's A-P multiplier ratios by (3.8) and in step 3, obtain  $SI_k$  by averaging each column scores by (3.9).

$$\begin{aligned}
 & \text{subject to } \begin{aligned} & \text{Max}_{u_r} \theta_j = \sum_{r=1}^s \mu_{r,j} y_{r,j} \\ & \sum_{i=1}^m v_{i,j} x_{ij} = 1 \\ & \sum_{r=1}^s \mu_{r,j} y_{r,j} - \sum_{i=1}^m v_{i,j} x_{i,j} \leq 0, \quad j = 1, \dots, n \end{aligned} \tag{3.8}
 \end{aligned}$$

$$\frac{v_1}{v_2} = \frac{v_{1,k}}{v_{2,k}}, \dots, \frac{v_1}{v_m} = \frac{v_{1,k}}{v_{m,k}},$$

$$\frac{\mu_1}{\mu_2} = \frac{\mu_{1,k}}{\mu_{2,k}}, \dots, \frac{\mu_1}{\mu_s} = \frac{\mu_{1,k}}{\mu_{s,k}},$$

$$\mu_{r,j}, v_{i,j} \geq 0, \quad \forall r \text{ and } i$$

where,  $v_{i,k}$  = DMU  $k$ 's A-P multiplier for input  $i$

$\mu_{r,k}$  = DMU  $k$ 's A-P multiplier for output  $r$

$$SI_k = \frac{1}{n} \sum_{j=1}^n E_{j,k} \quad (3.9)$$

**Table 3.6 Results comparison of SI, A-P and cross-efficiency scores (FMS data)**

DMU	1	2	3	4	5	6	7	8	9	10	11	12
1	1.000	0.915	1.000	0.836	0.714	0.607	<b>0.943</b>	1.000	0.359	0.764	1.000	1.000
2	0.956	1.000	0.959	0.881	0.686	0.601	0.897	0.977	0.367	0.704	0.966	0.984
3	0.919	0.606	0.982	0.752	0.862	0.774	0.925	0.949	0.422	0.904	0.924	0.953
4	0.910	0.915	0.927	1.000	0.862	0.801	0.891	1.000	0.482	0.956	0.896	1.000
5	0.956	0.794	1.000	0.881	1.000	0.815	0.943	1.000	0.435	1.000	0.927	1.000
6	0.825	0.438	0.932	0.744	0.547	1.000	0.943	0.962	0.698	0.951	0.954	0.967
7	0.897	0.459	1.000	0.748	0.628	0.973	1.000	1.000	0.643	1.000	1.000	1.000
8	0.916	0.608	0.927	0.777	0.584	0.687	0.917	0.961	0.450	0.860	0.945	0.950
9	0.482	0.315	0.422	0.632	0.010	<b>0.973</b>	0.659	0.753	1.000	0.849	0.672	0.724
10	0.772	0.427	0.760	0.642	0.454	0.679	0.822	0.833	0.468	0.954	0.783	0.795
11	0.956	0.665	0.977	0.719	0.625	0.590	0.923	0.951	0.355	0.714	0.983	0.953
12	0.760	0.608	0.801	0.687	0.727	0.625	0.748	0.794	0.347	0.720	0.759	0.801
<b>SI</b>	<b>0.862</b>	<b>0.646</b>	<b>0.891</b>	<b>0.775</b>	<b>0.642</b>	<b>0.760</b>	<b>0.884</b>	<b>0.932</b>	<b>0.502</b>	<b>0.865</b>	<b>0.901</b>	<b>0.927</b>
rank	6	3	9	5	2	4	8	12	1	7	10	11
<b>CE</b>	<b>0.848</b>	<b>0.839</b>	<b>0.777</b>	<b>0.844</b>	<b>0.867</b>	<b>0.727</b>	<b>0.758</b>	<b>0.724</b>	<b>0.564</b>	<b>0.618</b>	<b>0.747</b>	<b>0.667</b>
rank	2	4	5	3	1	8	6	9	12	11	7	10
<b>A-P</b>	<b>1.045</b>	<b>1.093</b>	<b>0.982</b>	<b>1.134</b>	<b>1.160</b>	<b>1.028</b>	<b>1.06</b>	<b>0.961</b>	<b>1.432</b>	<b>0.953</b>	<b>0.983</b>	<b>0.801</b>
rank	6	4	8	3	2	7	5	10	1	11	8	12

Table 3.6 shows the results of SI, A-P and cross-efficiency scores in FMS example.

Detailed results of SI calculation in car selection example are also shown in Table A.3 in the appendix.

The *SI* score under the column of DMU 1 represents the average of all DMU's efficiency scores when they use DMU 1's A-P multiplier ratios using (3.8) and (3.9). All the other DMU's *SI* scores are calculated in the similar manner. The high value of  $SI_k$  indicates that many other DMUs are competitive with DMU  $j$ 's A-P weight vectors, which represents DMU  $j$ 's weight vectors are not unusual. On the other hand, the low value of  $SI_k$  indicates that many other DMUs are not competitive (do not favor) with DMU  $j$ 's weight vector, which represents DMU  $j$ 's weight vectors are specialized.

When we see the results in Table 3.6, DMU 9 (0.502) shows the smallest *SI* score that represent DMU 9 is the most specialized, which ranked 1<sup>st</sup> in *SI* score. Then DMU 5 (0.652), DMU 2 (0.646), ... have specialized weights. When we see the ranking of A-P model, most of the specialized DMUs are assigned with high ranking but the order is not exactly the same. This is due to the fact that DMU  $j$ 's A-P score is determined by the ratio to that of the 2<sup>nd</sup> follower. Therefore, even though DMU  $j$  is the most specialized in *SI* score, if there is the near 2<sup>nd</sup> follower in that weight, DMU  $j$  will typically not be ranked 1<sup>st</sup>.

For example, A-P score of DMU 6 is measured by  $(1/0.973 = 1.028)$  and that of DMU 7 is measured by  $(1/0.943=1.06)$ . DMU 6 is more specialized than DMU 7 in *SI* score, but DMU 6 has 2<sup>nd</sup> follower (DMU 7 : 0.973) and DMU 7 has 2<sup>nd</sup> follower (DMU 1 : 0.943) which makes DMU 7 higher ranked than DMU 6.

Also, we can see clear examples in the results in Table A.3 in the appendix. Among the CCR efficient DMUs, DMU 1 (0.458), DMU 21(0.572), DMU 28 (0.575), DMU 11 (0.619) have the specialized weights in *SI* order, but the A-P ranking of DMU 28 is not high (12<sup>th</sup>) compared with the other DMUs. It is because DMU 28 has a nearer 2<sup>nd</sup> follower of DMU 24(0.959) than other DMUs.

Table 3.7 shows the ranking comparison of A-P model and cross-evaluation with respect to *SI* scores, where rank\*\* represents the rank of *SI* scores. That is, the most specialized DMU which has the lowest *SI* score is ranked 1<sup>st</sup>.

In all 3 examples below, the upper rankers of A-P model are one of the most specialized DMUs. Also in two cases (application 1 and 3), the 1<sup>st</sup> ranker of cross-evaluation is one of the most specialized DMUs, which ranked 2<sup>nd</sup> in *SI*. And this contradicts the traditional belief on the characteristics of the 1<sup>st</sup> ranker in cross-evaluation, which is discussed in next section with more detail.

**Table 3.7 Comparison of A-P and cross-evaluation with respect to *SI***

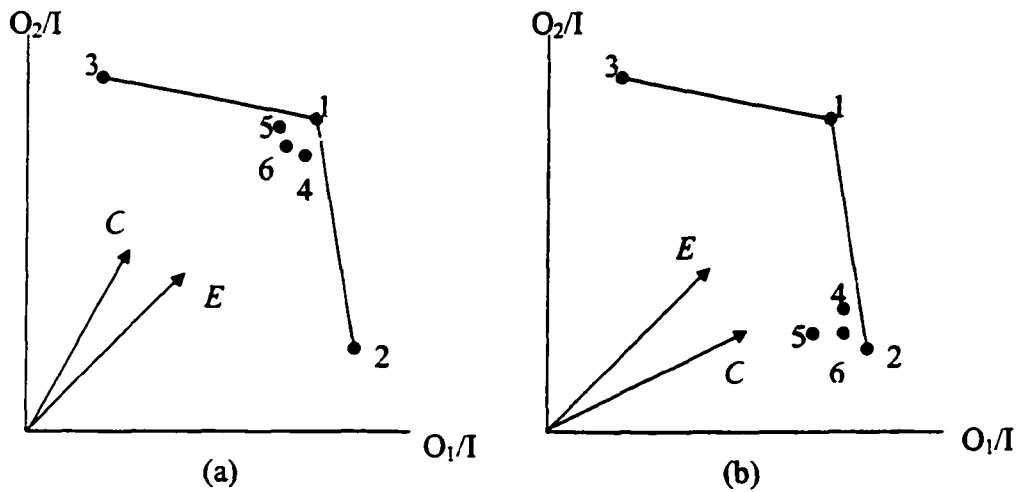
		Application 1 (n=12)			Application 2 (n=28)			Application 3 (n=6)		
		DMU	rank**		DMU	rank**		DMU	rank**	
<b>A-P</b>	1 <sup>st</sup>	9	(1)		1 <sup>st</sup>	1	(1)	1 <sup>st</sup>	2	(1)
	2 <sup>nd</sup>	5	(2)		2 <sup>nd</sup>	21	(3)	2 <sup>nd</sup>	5	(2)
	3 <sup>rd</sup>	4	(5)		3 <sup>rd</sup>	25	(7)	3 <sup>rd</sup>	4	(3)
<b>CE</b>	1 <sup>st</sup>	5	(2)		1 <sup>st</sup>	6	(13)	1 <sup>st</sup>	5	(2)
	2 <sup>nd</sup>	1	(6)		2 <sup>nd</sup>	12	(10)	2 <sup>nd</sup>	6	(5)
	3 <sup>rd</sup>	4	(5)		3 <sup>rd</sup>	3	(20)	3 <sup>rd</sup>	4	(3)

### 3.5 Identification of all-round (overall) performer

#### (1) Examination of the primal claim on the 1<sup>st</sup> ranker

In this section, we tried to identify the characteristics of cross-efficiency evaluation in the light of following two aspects.

- 1) Is the 1<sup>st</sup> ranker in cross-evaluation always the winner with many competitors?
- 2) How different are the ranking criteria of cross-efficiency evaluation (to find an all-round performer) from applying equal input and equal output weights (to find a general performer defined below)?



**Figure 3.3 The criteria of cross-efficiency evaluation**

Figure 3.3 (b) simply shows that popular conception that cross-efficiency evaluation selects the winner with many competitors is not always true. In both cases of Figure 3.3 (a) and (b), DMU 1 is the 1<sup>st</sup> ranker by cross-efficiency evaluation. However in Figure 3.3 (b), DMU 1 does not appear to be the winner with many competitors. Table 3.8 (a), (b) show the results comparison of maverick index  $M_k$  and  $A_k$  score of each case of Figure 3.3 (a), (b).

**Table 3.8 Results comparison of  $M_k$  and  $A_k$**

(a) Figure 3.3 (a)							(b) Figure 3.3 (b)						
DMU	1	2	3	4	5	6	DMU	1	2	3	4	5	6
$M_k$	0.029	1.453	0.397	0.039	0.043	0.031	$M_k$	0.029	0.438	1.214	0.372	0.392	0.400
rank	1	6	5	3	4	2	rank	1	5	6	2	3	4
$A_k$	0.864	0.803	0.810	0.870	0.864	0.864	$A_k$	0.543	0.828	0.479	0.860	0.860	0.860

In both cases, the result of  $M_k$  represents that DMU 1 is the least maverick in the sample but it doesn't correspond to DMU 1's distinctiveness, which is clearly seen in Figure 3.3 (b). In order to quantify the distinctiveness, we suggest that the  $SI$  score or  $A_k$  score from

the cross-efficiency matrix is more appropriate than the maverick index. From  $A_k$  score, we can see that DMUs 2, 3 are distinctive in case of Figure 3.3 (a) and DMUs 1, 3 are distinctive in case of Figure 3.3 (b).

This fact also can be found from the result in Table 3.4, DMU 5 shows the second lowest  $SI$  score and thus considered as one of specialized DMUs but cross-efficiency evaluation ranked as 1<sup>st</sup>. Therefore, we can see that the 1<sup>st</sup> ranker is not always among the least specialized DMUs in the sample.

## (2) Empirical behavior

Actually, cross-efficiency has been considered a powerful ranking scheme since it can make a unique ranking without any given weights. In this sense, Doyle et al [12] also mentioned that cross-efficiency can be a useful tool for lazy decision makers.

On the other hand, when we don't know the weights or don't have any weight priorities, it is also very simple and natural idea to select a DMU as 1<sup>st</sup> ranker that would have the highest probability of obtaining the highest efficiency score for a randomly selected set of weights. This could be approximated by sampling over a grid over the entire space of possible weights. Since this is not practical for problems with a high number of inputs and outputs, an alternative approach is to evaluate each DMU using equal input weights and output weights. Typically the DMU that achieves the highest score using equal weights would tend to be well-suited for obtaining the highest rank for a wide range of weights. And this DMU can be found by performing CCR model with weight restrictions of both equal input weights and equal output weights (3.10).

$$v_i = v_{i+1} \quad (i = 1 \cdots m - 1), \quad \mu_r = \mu_{r+1} \quad (r = 1 \cdots s - 1) \quad (3.10)$$

Here we introduce another term “general performer” which is the 1<sup>st</sup> ranker with weight restrictions (3.10) to discriminate from the less rigorous “all-round performer” (overall performer) in cross-efficiency evaluation.

To explain the above idea, we performed an empirical study on FMS and car selection problems. Each result is shown in appendix A.4 and A.5. Random weight ratios are repeatedly generated for FMS ( $N = 20$ ) and car selection problems ( $N = 30$ ) and each row in Table A.4 and A.5 shows the efficiency scores on these random weight ratios. The final three rows represent mean scores under random weights, cross-efficiency scores and results with restrictions (3.10).

In the result of A.4, we can easily see that DMU 5 will have the largest average efficiency score and be the 1<sup>st</sup> ranker. Even though we showed the result of just 20 and 30 iterations in each problem, the ranking results are very similar to those with weight restrictions (3.10). For these problems, it clearly appears that additional iterations will maintain a ranking result that will be exactly same with that with weight restrictions (3.10).

In each problem, DMU 5 and 6 are the 1<sup>st</sup> rankers respectively and we call them “general performer” which are also the 1<sup>st</sup> rankers in cross-efficiency. Another interesting thing is in both cases, the rankings of cross-evaluation are similar to those with weight restrictions (3.10).

We also compared the results in 4 other application problems in [21], [26], [42], [12]. The same 1<sup>st</sup> ranker is found in the result of 3 problems [26], [42], [12] and different 1<sup>st</sup> ranker was found in 1 problem [21]. Moreover, the ranking results are not much different in all 4 cases. Then how different is the ranking criteria of cross-efficiency with that with weight restrictions (3.10)? Since cross-efficiency uses almost fixed weights in many of



multiple-input, multiple-output cases, which will be discussed in chapter 4 in detail, the difference can be found in the direction of weight vectors.

In Figure 3.3 (a) and (b),  $E$  represents the direction of weight vectors of equal weights and  $C$  represents the direction of weight vectors actually applied in cross-efficiency evaluation. The direction of cross-efficiency weight vector is changed along with changed DMUs. Actually, ratios of cross-efficiency multipliers of DMU  $j$  indicate the most favorable direction for each DMU among CCR multipliers (in the sense that it can minimize all the other DMU's virtual output). Since cross-efficiency evaluation averages the results of each DMUs in sample, the resulting directions of weight vectors are closer to those which many DMUs favor. In Figure 3.3 (b), the result direction of weight vectors is closer to DMU 2 by averaging process that gives the maverick DMU 3 a low efficiency score.

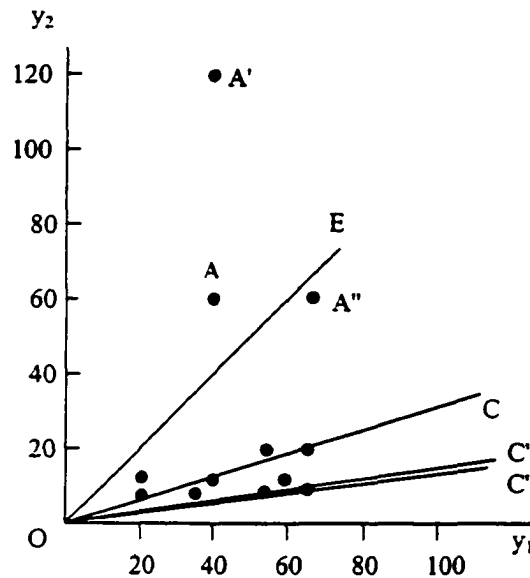
### (3) Unexpected result

Since the final weight direction in cross-efficiency evaluation is determined by the averaging process of each DMU's multipliers, it is necessary to select sample DMUs carefully. According to the sample selection, cross-efficiency evaluation may sometimes yield unexpected results.

Figure 3.4 shows the importance of sample selection in cross-efficiency evaluation. For simplicity, we assume that we evaluate the performance of 10 baseball players using one input, two outputs, i.e. number of hits ( $y_1$ ) and number of homeruns ( $y_2$ ) made in a year.

Also we assume that player A was generally expected as one of the best players in that year since he recorded the most number of homeruns ( $y_2$ ) with about an average number of

hits ( $y_1$ ). Among 10 players, we compared 3 cases with changing player A to player A' and A". In each case, all the other players are remained same. That is  $(y_1, y_2) = A(40, 60)$ ,  $A'(40, 120)$ ,  $A''(65,60)$ .



**Figure 3.4 The importance of sample selection**

$E$  represents the direction of weight vectors of equal weights and  $C, C', C''$  represent the direction of weight vectors actually applied in cross-efficiency evaluation in each case of player A, A', A" respectively. By cross-efficiency evaluation, player A and A' appeared just an average performer (ranked 5<sup>th</sup>) since much higher weights are applied to  $y_1$  than  $y_2$ . Of course, if we applied equal weights to variables or higher weights to  $y_2$ , player A and A' will be appeared easily the best performer.

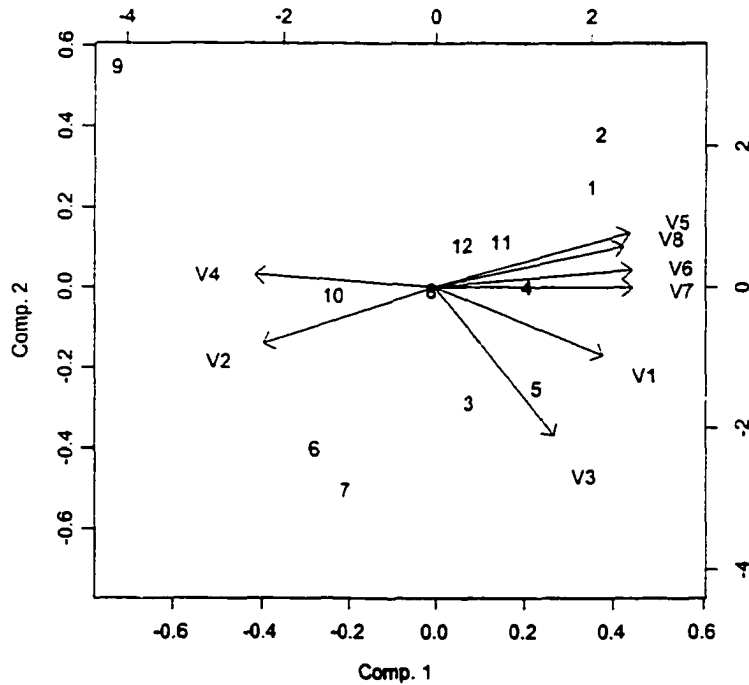
However player A" is appeared the 1<sup>st</sup> ranker by cross-efficiency evaluation. In this case, it is unnatural to consider player A" as the winner with many competitors.

### 3.6 Biplot

We suggest a biplot of FMS data, which facilitates the comparison of characteristics of each model visually. The biplot is constructed from the correlations of the following variables. All variables are made to have maximizing criteria.

$$(V_1, V_2, \dots, V_n) = \left( \frac{y_1}{x_1}, \frac{y_2}{x_1}, \dots, \frac{y_s}{x_1}, \frac{y_1}{x_2}, \frac{y_2}{x_2}, \dots, \frac{y_s}{x_2}, \dots, \frac{y_1}{x_m}, \frac{y_2}{x_m}, \dots, \frac{y_s}{x_m} \right) \quad (3.11)$$

Biplot permits the visual inspection of one DMU relative to another and the relative importance of each of the two variables to the position of any DMU.



**Figure 3.5 Biplot of the FMS data**

From above variable relation (3.11) we have 8 variables in FMS example. That is,

$$(V_1, V_2, \dots, V_8) = \left( \frac{y_1}{x_1}, \frac{y_2}{x_1}, \dots, \frac{y_4}{x_2} \right). \text{ And the proportion of variance explained from this}$$

biplot is 88.1% of total variance due to two principal components. The biplot in Figure 3.5 shows well the relative position of each DMU to changed variables.

This biplot can satisfy our interest in developing a visual picture of the relative position among DMUs. It is interesting to see that all of the CCR efficient DMUs (1, 2, 4, 5, 6, 7, 9) are positioned relatively at the end of each variable and all of CCR inefficient DMUs (3, 8, 10, 11, 12) are positioned near the origin of each variable. Also, the specialized DMU can be seen visually from the biplot. In the biplot, DMU 9 is located upper-left corner far apart from all the other CCR efficient DMUs (1, 2, 4, 5, 6 and 7).

Biplot in Figure 3.6 includes another variable, which represents cross-efficiency score with fixed weights. That is

$$(V_1, V_2, \dots, V_n, V_{n+1}) \quad (3.12)$$

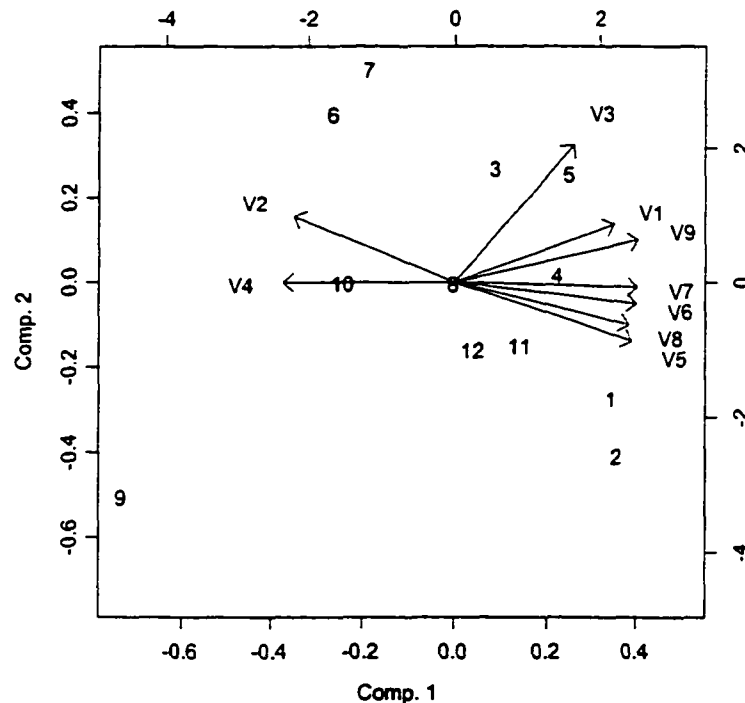
where,  $V_1, V_2, \dots, V_n$  are the same as (3.11)

$$V_{n+1} = CE^* = \frac{\sum_{r=1}^s y_{r,j} \bar{\mu}_r}{\sum_{i=1}^m x_{i,j} \bar{v}_i}$$

In Figure 3.6,  $V_{n+1} = V_9$ , which represents the direction of cross-efficiency evaluation with fixed weights. From the biplot with additional variable, we can see more clearly that DMUs 5, 1, 4, 2, 3 are positioned as higher rankers, DMUs 7, 6 are middle rankers and DMU 9 should be the lowest ranker among CCR efficient DMUs when we use cross-evaluation.

The applied direction of fixed weight (V9) is almost opposite of V4 that is the most favorable weight direction of DMU 9. And even though this biplot cannot display total

variance (proportion of variance is 89.1% in this case) these look to pretty well coincide with the calculation results.



**Figure 3.6 Direction of the fixed weights in cross-efficiency evaluation**

### 3.7 Conclusions

A-P model and cross-efficiency evaluation often makes different ranking results in many applications. Therefore, it is necessary to know each model's ranking characteristics to avoid misleading decision.

The contributions of this chapter are as follows.

First, we developed specialization index (SI), which enables to identify specialized DMUs in A-P model and proposed using  $A_t$  score as a replacement of Maverick index  $M_t$  in cross-efficiency evaluation. Second, we showed that the primal claims on the characteristics of 1<sup>st</sup>

ranker in cross-efficiency evaluation are not correct. Third, we proposed a methodology for developing a biplot, which facilitates the visual comparison of DMUs in models with more than 3 inputs and outputs. Empirical studies are performed to compare the 1<sup>st</sup> ranker in cross-evaluation with that under restriction of equal input, output weights along with the explanation of simple case in which cross-efficiency makes unexpected ranking result.

## **CHAPTER 4. EXTENSIONS ON THE FIXED WEIGHTING NATURE OF CROSS-EFFICIENCY EVALUATION**

### **4.1 Introduction**

In the earlier paper [4], Anderson et al. demonstrated that in a single-input, multiple-output constant returns to scale model with input orientation, cross-efficiency evaluation in effect applies implicitly fixed weights to each and every DMU, which is a weighted average of the weights used by all of the DMUs in the sample.

They also stated that

- 1) The common set of weights also exists in the multiple-input, single-output constant returns to scale model with output orientation.
- 2) The multiple-input, multiple-output models do not exhibit this fixed weighting phenomena because of the inability to normalize the weights.

Based on their ideas, in this chapter we made an extension to the multiple-inputs, multiple-outputs constant returns to scale with input/output orientation to show that

- 1) The cross-evaluations do not use the column average of cross-efficiency multipliers as fixed weights exactly in single-input, multiple-output case when input is not unified as 1 and also multiple-input, multiple-output situation.
- 2) Even though cross-evaluations don't use the exact fixed weights, the column average value of cross-efficiency multipliers can be considered as the fixed weights to each variable without much difference in many cases.
- 3) The above difference of DMU  $j$  is caused by the DMU  $j'$  that favors far different weights and thus DMU  $j$  can not dominate each of DMU  $j'$  when using weight vectors of DMU  $j'$ . Therefore, a certain CCR-efficient DMU  $j$ , which is maverick in the

sample and cannot dominate many DMUs, will show rather large difference than CCR-efficient DMU  $j'$ , which is less maverick and can dominate many DMUs.

The rest of this chapter is organized as follows.

In section 4.2, fixed weighting nature of cross-efficiency evaluation in single-input, multiple-output situation is presented. In this section, we showed the fact that when the input values of all DMUs are not unified 1, cross-efficiency scores are not exactly same with those under fixed weighting scheme. As an extension of single-input, multiple-output situation, in section 4.3, we developed an equation, which shows an efficiency score under fixed weighting scheme in multiple-input, multiple-output situation. And in section 4.4, we analyzed the difference between real cross-efficiency score and that under fixed weighting scheme in multiple-input, multiple-output situation analytically and empirically. Finally, conclusions are provided in section 4.5.

## 4.2 Single-input, Multiple outputs case

### 4.2.1 The case that each DMU's input value is unified to 1

For illustration purpose, we will begin with brief explanation of Anderson et al [4]'s work and example. Cross-efficiency score of DMU  $k$  is calculated as equation (4.1)

$$CE_k = \frac{1}{n} \left[ \frac{\sum_{j=1}^n \frac{\sum_{r=1}^s \mu_{r,j} y_{r,k}}{\sum_{i=1}^m v_{i,j} x_{i,k}}}{\sum_{i=1}^m v_{i,j} x_{i,k}} \right] \quad (4.1)$$

In the single-input, multiple-output case (when  $x_{i,k} = 1$ ), it becomes



$$\begin{aligned}
CE_k &= \frac{1}{n} \left[ \sum_{j=1}^n \frac{\sum_{r=1}^s \mu_{r,j} y_{r,k}}{\sum_{i=1}^m v_{i,j} x_{i,k}} \right] = \frac{1}{n} \left[ \sum_{j=1}^n \frac{\sum_{r=1}^s \mu_{r,j} y_{r,k}}{\sum_{i=1}^m v_{1,j} x_{1,k}} \right] = \frac{1}{nx_{1,k}} \left[ \sum_{j=1}^n \sum_{r=1}^s \frac{\mu_{r,j} y_{r,k}}{v_{1,j}} \right] \\
&= \frac{1}{nx_{1,k}} \left[ \sum_{r=1}^s y_{r,k} \sum_{j=1}^n \left( \frac{\mu_{r,j}}{v_{1,j}} \right) \right] = \frac{\sum_{r=1}^s y_{r,k} \left[ \frac{1}{n} \sum_{j=1}^n \left( \frac{\mu_{r,j}}{v_{1,j}} \right) \right]}{x_{1,k}} \tag{4.2}
\end{aligned}$$

As a result, the multipliers  $\left[ \frac{1}{n} \sum_{j=1}^n \left( \frac{\mu_{r,j}}{v_{1,j}} \right) \right]$  for each output  $y_{r,k}$  are independent of DMU

$k$ . That is, in the following example, the weights for each output are 0.0323175, 0.05257 respectively. And they also said that these weight results match those obtained using the standard column average method to four decimal places of accuracy.

However the weights actually applied for each output in cross efficiency evaluation (single- input, multiple-outputs case with all input values are unified to 1) is exactly same with the value  $\frac{\bar{\mu}_{r,j}}{\bar{v}}$ . That is column average of each output multipliers / column average of input multipliers. That is, by the first constraint of cross-efficiency evaluation, input-multipliers for all DMUs should be the same and therefore the column mean of input-multipliers is also the same with input multiplier for each DMU.

In the single-input, multiple-output case (when  $x_{1,j} = 1$ ), the above equation (4.2) can be changed to equation (4.3) as follows.

1. By the first constraint of cross-efficiency evaluation

$$\sum_{i=1}^m \left( v_{i,j} \sum_{\substack{j=1 \\ j \neq j_0}}^n x_{i,j} \right) = (n-1) v_{1,j} = 1, \quad \text{thus} \quad v_{1,j} = \frac{1}{n-1}, \quad \forall j$$

(For example, in the following example  $v_{1,j} = \frac{1}{6-1} = 0.2$  for all  $j$ ).

2. Also  $\bar{v} = \frac{1}{n} \sum_{j=1}^n v_{1,j} = \frac{1}{n} \times \frac{n}{n-1} = \frac{1}{n-1} = v_{1,j}$

3. Therefore

$$\begin{aligned} CE_k &= \frac{1}{n} \left[ \frac{\sum_{r=1}^s \mu_{r,j} y_{r,k}}{\sum_{i=1}^m v_{i,j} x_{i,k}} \right] = \frac{1}{n} \left[ \frac{\sum_{r=1}^s \mu_{r,j} y_{r,k}}{\sum_{i=1}^m v_{1,j} x_{1,k}} \right] = \frac{1}{nx_{1,k} v_{1,j}} \left[ \sum_{j=1}^n \sum_{r=1}^s \mu_{r,j} y_{r,k} \right] \\ &= \frac{1}{n\bar{v}} \left[ \sum_{r=1}^s \sum_{j=1}^n \mu_{r,j} y_{r,k} \right] = \sum_{r=1}^s y_{r,k} \left[ \frac{\sum_{j=1}^n \mu_{r,j} / n}{\bar{v}} \right] = \sum_{r=1}^s y_{r,k} \left[ \frac{\bar{\mu}_r}{\bar{v}} \right] \end{aligned} \quad (4.3)$$

where,  $\bar{\mu}_r$  is the average value of each output weights,

$v_{1,j} = \bar{v}$  (for all  $j$ ) is the average value of input weights.

After all, the multipliers  $\left[ \frac{1}{n} \sum_{j=1}^n \left( \frac{\mu_{r,j}}{v_{1,j}} \right) \right]$  for each output  $y_{r,k}$  can be expressed as  $\left[ \frac{\bar{\mu}_r}{\bar{v}} \right]$  which

are independent of DMU  $k$ . We will show this by the following example 4.1.

Table 4.1 shows an example 4.1 data (left table), CCR efficiency score (2<sup>nd</sup> column in right table) and cross-efficiency multipliers (3<sup>rd</sup> ~ 5<sup>th</sup> column in right table), and Table 4.2 shows a cross-efficiency matrix and efficiency scores represented as mean in the last row.

**Table 4.1 Example 4.1 data and cross efficiency weights**

Data				CEM Multiplier - Input Oriented				
DMU	$x$	$y_1$	$y_2$	DMU	CCR	$\nu_1$	$\mu_1$	$\mu_2$
1	1	10.7	12.0	1	1	0.2	0.0015	0.0153
2	1	11.6	2.5	2	1	0.2	0.0172	0
3	1	2.8	12.8	3	1	0.2	0	0.0156
4	1	10.5	11.6	4	0.9799	0.2	0.0169	0.0016
5	1	10.1	11.8	5	0.9801	0.2	0.0015	0.0153
6	1	10.2	11.5	6	0.9579	0.2	0.0015	0.0153
mean						0.2	0.0065	0.0105

**Table 4.2 Cross efficiency results for example 4.1**

DMU	1	2	3	4	5	6
1	1.000	0.281	1.000	0.968	0.980	0.958
2	0.922	1.000	0.241	0.905	0.871	0.879
3	0.938	0.195	1.000	0.906	0.922	0.898
4	1.000	1.000	0.339	0.980	0.948	0.954
5	1.000	0.281	1.000	0.968	0.980	0.958
6	1.000	0.281	1.000	0.968	0.980	0.958
mean	0.97664	0.50631	0.76339	0.94915	0.94673	0.93419

From the above equation (4.3), cross-efficiency scores of DMU 1 and 2 are calculated as follows.

$$\begin{aligned}
 CE_1 &= \frac{1}{6} \times \frac{1}{0.2} [(\mu_{11}y_{11} + \mu_{21}y_{21}) + (\mu_{12}y_{11} + \mu_{22}y_{21}) + \dots + (\mu_{16}y_{11} + \mu_{26}y_{21})] \\
 &= \frac{1}{6} \times \frac{1}{0.2} [y_{11}(\mu_{11} + \mu_{12} + \dots + \mu_{16}) + y_{21}(\mu_{21} + \mu_{22} + \dots + \mu_{26})] \\
 &= \frac{1}{\bar{\nu}} [y_{11} \bar{\mu}_1 + y_{21} \bar{\mu}_2] = \frac{1}{0.2} (y_{11} (0.0064635) + y_{21} (0.010514)) \\
 &= y_{11} (0.0323175) + y_{21} (0.052570) = 0.97664
 \end{aligned}$$

$$\begin{aligned}
 CE_2 &= \frac{1}{1.2} [y_{12}(\mu_{11} + \mu_{12} + \dots + \mu_{16}) + y_{22}(\mu_{21} + \mu_{22} + \dots + \mu_{26})] \\
 &= \frac{1}{\bar{\nu}} [y_{12} \bar{\mu}_1 + y_{22} \bar{\mu}_2] = \frac{1}{0.2} (y_{12} (0.0064635) + y_{22} (0.010514))
 \end{aligned}$$

$$= y_{12} (0.0323175) + y_{22} (0.052570) = 0.50631$$

As a conclusion in the case that single-input, multiple-outputs with all input values are equal to 1, cross-evaluation uses the fixed weights, which are exactly same as the column mean of the multipliers.

#### 4.2.2 The case that each DMU's input value is not unified to 1

However unlike the above example, if the input variables are not unified to 1 and have different values then above equation (4.3) should be modified as equation (4.4).

Because input variables are not unified to 1, the input multipliers and input values  $x_{1,k}$  in the cross-efficiency evaluation have different values and cannot be extracted out of the bracket.

Therefore we cannot get the fixed weights as above example and the final cross efficiency score also be different with above result.

$$\begin{aligned}
 CE_k &= \frac{1}{n} \left[ \sum_{j=1}^n \frac{\sum_{r=1}^s \mu_{r,j} y_{r,k}}{\sum_{i=1}^m v_{i,j} x_{i,k}} \right] = \frac{1}{n} \left[ \sum_{j=1}^n \frac{\sum_{r=1}^s \mu_{r,j} y_{r,k}}{\sum_{i=1}^m v_{1,j} x_{1,k}} \right] \\
 &= \frac{1}{n} \left[ \sum_{r=1}^s y_{r,k} \sum_{j=1}^n \left( \frac{\mu_{r,j}}{v_{1,j} x_{1,k}} \right) \right] = \sum_{r=1}^s y_{r,k} \left[ \frac{1}{n} \sum_{j=1}^n \left( \frac{\mu_{r,j}}{v_{1,j} x_{1,k}} \right) \right] \quad (4.4)
 \end{aligned}$$

Table 4.3 shows an example 4.2 data (left table), CCR efficiency score (2<sup>nd</sup> column in right table) and cross-efficiency multipliers (3<sup>rd</sup> ~5<sup>th</sup> column in right table), and Table 4.4 shows a cross-efficiency matrix and efficiency scores as represented as mean in the second last row.

**Table 4.3 Example 4.2 data and cross efficiency weights**

Data				CEM Multiplier - Input Oriented				
DMU	$x$	$y_1$	$y_2$	DMU	CCR	$v_1$	$\mu_1$	$\mu_2$
1	1	7	7	1	1	0.0345	0.0031	0.0018
2	5	50	10	2	1	0.0400	0.0040	0.0000
3	5	15	45	3	1	0.0400	0.0000	0.0044
4	8	48	48	4	0.8571	0.0455	0.0041	0.0024
5	10	50	70	5	0.9048	0.0500	0.0024	0.0048
6	1	5	8	6	1	0.0500	0.0024	0.0048
				mean		0.04332	0.00265	0.00304

**Table 4.4 Cross efficiency results for example 4.2**

DMU	1	2	3	4	5	6
1	1.000	1.000	0.750	0.857	0.821	0.875
2	0.700	1.000	0.300	0.600	0.500	0.500
3	0.778	0.222	1.000	0.667	0.778	0.889
4	1.000	1.000	0.750	0.857	0.821	0.875
5	1.000	0.667	1.000	0.857	0.905	1.000
6	1.000	0.667	1.000	0.857	0.905	1.000
mean	0.91294	0.75925	0.79997	0.78252	0.78834	0.85645
	0.91962	0.75206	0.81538	0.78824	0.79729	0.86750

In this case, input multipliers are not equal to each other. Therefore we cannot extract out the input multipliers from the bracket in equation (4.4) so that we cannot find the fixed weights for each output. Even though we cannot find the common fixed weights in this case, column means of multiplier values serve as a good indicator in this example too.

The last row of Table 4.4 represented as  $CE^*$  is calculated using the column mean as the fixed common weights to each variable (4.5).

$$CE^* = \frac{\sum_{r=1}^s y_{r,j} \bar{\mu}_r}{x_{i,j} \bar{v}} \quad (4.5)$$

where,  $\bar{v}$ ,  $\bar{\mu}_r$  are the mean values of the input and output multipliers, respectively.

It is also interesting to note that the values from those two methods are very similar to each other. But the differences of result between the two methods are not the same.

Until this we have examined the single-input, multiple-outputs case, and from next section we will expand this to the multiple-inputs, multiple-outputs case to see the characteristics of fixed weighting nature in cross-efficiency evaluation.

### 4.3 Multiple-inputs, multiple-outputs case

Like the single-input, multiple-outputs case (if  $x_{i,k}$  is not all changed to 1), cross-efficiency evaluation doesn't use each column mean of multiplier values as fixed weights in the multiple input, multiple-output case.

However even in multiple-inputs, multiple-outputs case, when the cross-efficiency evaluation uses the column means as fixed weights and we calculate the efficiency score by equation (4.6), the result is very similar to the true result especially for the high rankers in cross-efficiency evaluation.

$$CE^* = \frac{\sum_{r=1}^s y_{r,j} \bar{\mu}_r}{\sum_{i=1}^m x_{i,j} \bar{v}_i} \quad (4.6)$$

To confirm this we will compare two equations, one is an equation of cross-efficiency score and the other is the equation (4.6) using the column mean of multipliers as a fixed weight. From the comparison of two equations and application example, we will show the following.

- 1) The validity of using column means of multipliers as fixed weights in multiple-input, multiple-output cases.
- 2) To find the source to make those result to be different.
- 3) To examine the magnitude of differences of efficiency scores among DMUs.



$$\begin{aligned}
&= \frac{y_{1,k}(\mu_{1,1} + \mu_{1,2} + \dots + \mu_{1,j}) + y_{2,k}(\mu_{2,1} + \mu_{2,2} + \dots + \mu_{2,j}) + \dots + y_{s,k}(\mu_{s,1} + \mu_{s,2} + \dots + \mu_{s,j})}{x_{1,k}(v_{1,1} + v_{1,2} + \dots + v_{1,j}) + x_{2,k}(v_{2,1} + v_{2,2} + \dots + v_{2,j}) + \dots + x_{m,k}(v_{m,1} + v_{m,2} + \dots + v_{m,j})} \\
&= \frac{(y_{1,k}\mu_{1,1} + \dots + y_{s,k}\mu_{s,1}) + (y_{1,k}\mu_{1,2} + \dots + y_{s,k}\mu_{s,2}) + \dots + (y_{1,k}\mu_{1,j} + \dots + y_{s,k}\mu_{s,j})}{(x_{1,k}v_{1,1} + \dots + x_{m,k}v_{m,1}) + (x_{1,k}v_{1,2} + \dots + x_{m,k}v_{m,2}) + \dots + (x_{1,k}v_{1,j} + \dots + x_{m,k}v_{m,j})} \\
&= \frac{b(1) + b(2) + b(3) + \dots + b(j)}{a(1) + a(2) + a(3) + \dots + a(j)} \Big|_k \tag{4.8}
\end{aligned}$$

After all, it is clear from above two equation (4.7) and (4.8) that the results from those two methods are not same.

$$CE_k = \frac{1}{n} \left[ \frac{b(1)}{a(1)} + \frac{b(2)}{a(2)} + \frac{b(3)}{a(3)} + \dots + \frac{b(j)}{a(j)} \right]_k \tag{4.7}$$

$$CE_k^* = \left[ \frac{b(1) + b(2) + b(3) + \dots + b(j)}{a(1) + a(2) + a(3) + \dots + a(j)} \right]_k \tag{4.8}$$

After all, equation (4.7) means the real cross-efficiency score while equation (4.8) calculates the efficiency score assuming each DMU to use multiplier column mean values as the fixed weights.

where,  $a(j)$  = virtual input when we use DMU  $j$ 's input multipliers

$b(j)$  = virtual output when we use DMU  $j$ 's output multipliers

For convenience, we explain two equations (4.7) and (4.8) using example 4.1 data in section 4.2.1. The real cross-efficiency score of DMU 1 is calculated as equation (4.7)

$$\begin{aligned}
CE_1 &= \frac{1}{6} \times \left[ \frac{(10.7 \times 0.0015) + (12 \times 0.0153)}{1 \times 0.2} + \frac{(10.7 \times 0.0172) + (12 \times 0)}{1 \times 0.2} + \dots \right. \\
&\quad \left. \dots + \frac{(10.7 \times 0.0015) + (12 \times 0.0153)}{1 \times 0.2} \right] = 0.97664
\end{aligned}$$



where,  $a(j) = 1 \times 0.2 \quad \forall j$

$$b(1) = (10.7 \times 0.0015) + (12 \times 0.0153), \quad b(2) = (10.7 \times 0.0172) + (12 \times 0), \dots$$

On the other hand, when we assume DMUs use the fixed weights as the average values of obtained multipliers, the efficiency score DMU  $k$  can be represented as equation (4.8), i.e.

$$\begin{aligned} CE_1^* &= \frac{10.7 \times \frac{1}{6} (0.0015 + 0.0172 + \dots + 0.0015) + 12 \times \frac{1}{6} (0.0153 + 0 + \dots + 0.0153)}{1 \times \frac{1}{6} (0.2 + 0.2 + \dots + 0.2)} \\ &= \frac{10.7 \times (0.0015 + 0.0172 + \dots + 0.0015) + 12 \times (0.0153 + 0 + \dots + 0.0153)}{1 \times (0.2 + 0.2 + \dots + 0.2)} \\ &= \frac{(10.7 \times 0.0015 + 12 \times 0.0153) + (10.7 \times 0.0172 + 12 \times 0) + \dots + (10.7 \times 0.0015 + 12 \times 0.0153)}{(1 \times 0.2) + (1 \times 0.2) + \dots + (1 \times 0.2)} \\ &= \left[ \frac{b(1) + b(2) + b(3) + \dots + b(j)}{a(1) + a(2) + a(3) + \dots + a(j)} \right]_1 = 0.97664 \end{aligned}$$

When we evaluate the DMU  $k$ , if most values of  $b(j)/a(j)$  ( $j = 1, \dots, n$ ) are close to the DMU  $k$ 's CCR efficiency score  $b(k)/a(k)$ , then the difference in results of above two equations should be very small. For example when we assume that DMU 1's CCR efficiency score is 1.0, then  $b(1)/a(1) = 1$ . If all the other values  $b(2)/a(2), \dots, b(j)/a(j)$  are all close to 1, above two equation values should be very similar.

To verify these ideas we will show the results using the application example in the next section. The purpose of deriving equation (4.8) is not to show that the results from two

equations (4.7) and (4.8) is the same but to show the characteristics of differences in the results according to DMUs.

#### 4.4 The analysis of differences between two equations

When we let DMU  $k$ 's CCR efficiency score as  $c_k$ , then we can say

$$\frac{b(1)}{a(1)} = c_k - \alpha_1, \dots, \frac{b(k)}{a(k)} = c_k, \dots, \frac{b(j)}{a(j)} = c_k - \alpha_j \quad (4.9)$$

(For example, DMU 3's CCR efficiency score in FMS data is  $c_3 = 0.982$ ,

and  $\frac{b(1)}{a(1)} = c_k - \alpha_1 = 0.982 - \alpha_1 = 0.630$ . Therefore  $\alpha_1 = 0.352$  )

Therefore equation (4.8) becomes

$$\begin{aligned} CE_k^* &= \left[ \frac{b(1) + b(2) + b(3) + \dots + b(j)}{a(1) + a(2) + a(3) + \dots + a(j)} \right]_k \\ &= \left[ \frac{\{a(1)c_k - a(1)\alpha_1\} + \dots + \{a(j)c_k - a(j)\alpha_j\}}{a(1) + a(2) + a(3) + \dots + a(j)} \right]_k \\ &= \left[ \frac{c_k(a(1) + \dots + a(j)) - (a(1)\alpha_1 + \dots + a(j)\alpha_j)}{a(1) + a(2) + a(3) + \dots + a(j)} \right]_k \\ &= \left[ c_k - \frac{a(1)\alpha_1 + \dots + a(j)\alpha_j}{a(1) + a(2) + a(3) + \dots + a(j)} \right]_k \end{aligned} \quad (4.10)$$

On the other hand, equation (4.7) becomes

$$\begin{aligned} CE_k &= \frac{1}{n} \left[ \frac{b(1)}{a(1)} + \frac{b(2)}{a(2)} + \frac{b(3)}{a(3)} + \dots + \frac{b(j)}{a(j)} \right]_k \\ &= \frac{1}{n} \left[ \frac{a(1)c_k - a(1)\alpha_1}{a(1)} + \dots + \frac{a(j)c_k - a(j)\alpha_j}{a(j)} \right]_k \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n} \left[ n c_k - (\alpha_1 + \dots + \alpha_j) \right]_k \\
&= \left[ c_k - \frac{\alpha_1 + \dots + \alpha_j}{n} \right]_k
\end{aligned} \tag{4.11}$$

Finally, when we subtract equation (4.10) from equation (4.11)

$$\begin{aligned}
CE_k - CE_k^* &= \left[ c_k - \frac{\alpha_1 + \dots + \alpha_j}{n} \right]_k - \left[ c_k - \frac{a(1)\alpha_1 + \dots + a(j)\alpha_j}{a(1) + \dots + a(j)} \right]_k \\
&= \left[ \frac{a(1)\alpha_1 + \dots + a(j)\alpha_j}{a(1) + \dots + a(j)} - \frac{\alpha(1) + \dots + \alpha(j)}{n} \right]_k \\
&= \alpha_1 \left[ \frac{a(1)}{a(1) + \dots + a(j)} - \frac{1}{n} \right] + \dots + \alpha_j \left[ \frac{a(j)}{a(1) + \dots + a(j)} - \frac{1}{n} \right]
\end{aligned} \tag{4.12}$$

In the single input, multi outputs case ( $x_{1k} = 1$ ,  $v_{1j} = 1/n - 1 = \text{constant } \forall j$ )

$$\frac{a(k)}{a(1) + \dots + a(j)} = \frac{x_{1k} v_{1k}}{x_{1k} v_{1,1} + \dots + x_{1k} v_{1j}} = \frac{1}{n}, \text{ therefore } CE_k - CE_k^* = 0 \tag{4.13}$$

Therefore, (4.13) verifies the result of (4.3). We cannot directly know from the equation (4.12) that the difference of two results is small. However we can define average difference, maximum difference of the results from two methods as follows (4.14), (4.15).

$$D = \sum_{k=1}^n d_k = \frac{1}{n} \sum_{k=1}^n (CE_k - CE_k^*) \tag{4.14}$$

$$\max(d_k) = \max (CE_k - CE_k^*) \tag{4.15}$$

In this section, we will examine the above derivations using FMS example. The data are shown in Table A.1 in the appendix and CCR efficiency scores, input / output cross efficiency multipliers are shown in Table 4.5. Also cross-efficiency scores for each DMU are

represented as  $CE$  in the Table 4.6 (second from last row), which is calculated by equation (4.7). Each DMU's self-efficiency score (CCR) is the corresponding diagonal elements of cross-efficiency matrix.

The final row ( $CE^*$ ) in Table 4.6 represents the efficiency score using the equation (4.8). It is surprising to see that these two results are not far different in general while some of the DMUs have very similar scores and the others show relatively big differences.

**Table 4.5 Cross efficiency multipliers of FMS data**

(FMS)	Efficiency (CCR)	Input weights		Output weights			
		$v_1$	$v_2$	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$
1	1	0.0024	0.0112	0.0023	0	0	0
2	1	0	0.0155	0.0018	0	0	0
3	0.9824	0.0083	0	0.0028	0	0.0016	0.0001
4	1	0.0006	0.0142	0	0	0.0029	0.0012
5	1	0	0.0153	0	0	0.0048	0
6	1	0.0078	0	0	0	0.0023	0.0016
7	1	0.0079	0	0	0.0009	0.0015	0.0007
8	0.9614	0.0070	0.0024	0.0019	0.0004	0	0.0011
9	1	0.0078	0	0	0	0	0.0016
10	0.9536	0.0069	0.0024	0	0.0022	0	0
11	0.9831	0.0087	0	0.0032	0	0	0.0005
12	0.8012	0.0072	0.0023	0.0020	0	0.0006	0.0014
mean		0.00539	0.00527	0.00117	0.00029	0.00115	0.00067

Among 12 DMUs, 7 DMUs (1, 2, 4, 5, 6, 7, and 9) are evaluated as CCR efficient (58.3%). DMU 5 is the 1<sup>st</sup> ranker and DMU 9 is ranked the last. Also it is interesting to see that the results of  $CE^*$  show very similar to  $CE$  through all DMUs while some of the DMUs have very close scores and the others show relatively big differences. DMU 5 (1<sup>st</sup> ranker) shows the smallest difference (0.0029) between  $CE$  and  $CE^*$  among the CCR efficient DMUs, and DMU 9 (12<sup>th</sup> ranker) shows the largest difference (0.2089) between those scores.

When we compare the DMUs 5 and 9 in detail, 6 DMUs (1, 2, 6, 7, 9 and 11) are the cause of difference in results of DMU 5 and 9 DMUs (1, 2, 3, 4, 5, 8, 10, 11 and 12) are the cause of difference in results of DMU 9. Especially we can see that DMUs 1, 2, 3, 4, 5 would cause large difference in results of DMU 9 while only DMU 9 cause large difference in results to DMU 5.

**Table 4.6 Cross efficiency results of FMS data**

DMU	1	2	3	4	5	6	7	8	9	10	11	12
1	1.000	1.000	0.630	0.725	0.742	0.322	0.384	0.615	0.094	0.371	0.813	0.593
2	0.969	1.000	0.500	0.634	0.638	0.214	0.260	0.481	0.058	0.264	0.699	0.502
3	1.000	0.959	0.982	0.927	1.000	0.932	1.000	0.927	0.422	0.760	0.977	0.801
4	0.957	1.000	0.758	1.000	1.000	0.434	0.482	0.610	0.190	0.418	0.703	0.732
5	0.899	0.915	0.728	0.895	1.000	0.293	0.358	0.475	0.004	0.294	0.624	0.705
6	0.601	0.597	0.764	0.794	0.804	1.000	0.970	0.685	1.000	0.678	0.585	0.617
7	0.630	0.599	0.807	0.820	0.857	1.000	1.000	0.732	1.000	0.795	0.610	0.637
8	1.000	0.977	0.949	1.000	1.000	0.962	1.000	0.961	0.753	0.833	0.951	0.794
9	0.359	0.367	0.422	0.482	0.435	0.698	0.643	0.450	1.000	0.468	0.355	0.347
10	0.764	0.704	0.904	0.956	1.000	0.951	1.000	0.860	0.849	0.953	0.714	0.720
11	1.000	0.966	0.924	0.896	0.927	0.954	1.000	0.945	0.672	0.783	0.983	0.759
12	1.000	0.985	0.953	1.000	1.000	0.968	1.000	0.950	0.724	0.796	0.953	0.801
CE	0.848	0.839	0.777	0.844	0.867	0.727	0.758	0.724	0.564	0.618	0.747	0.667
CE*	0.837	0.820	0.781	0.848	0.870	0.633	0.685	0.727	0.355	0.587	0.755	0.671

The average difference by (4.14) among all DMUs is  $D = 0.0386$ . Only three DMUs of 6, 7 and 9 have more than average values of  $d_k$  ( $d_9 = 0.2089$ ,  $d_6 = 0.0947$ ,  $d_7 = 0.0731$ ) and all the other DMUs have the value of  $d_k$  less than average.

The equation (4.12) can be confirmed by the following results which correspond with those calculated from final two rows in cross-efficiency matrix.

$$\begin{aligned}
 \text{(DMU 1)} \quad \frac{\alpha(1) + \dots + \alpha(j)}{n} &= \frac{1.82062}{12} = 0.15172 \\
 \frac{a(1)\alpha_1 + \dots + a(j)\alpha_j}{a(1) + \dots + a(j)} &= \frac{0.231633}{1.416456} = 0.16353 \\
 CE_k - CE_k^* &= 0.163530 - 0.15172 = 0.01181
 \end{aligned}$$

(DMU 9) shows the largest difference among DMUs between two results.

$$\begin{aligned}
 \frac{\alpha(1) + \dots + \alpha(j)}{n} &= \frac{5.233927}{12} = 0.436161 \\
 \frac{a(1)\alpha_1 + \dots + a(j)\alpha_j}{a(1) + \dots + a(j)} &= \frac{0.4796066}{0.743540} = 0.645031 \\
 CE_k - CE_k^* &= 0.645031 - 0.436161 = 0.20887
 \end{aligned}$$

We tried to calculate the differences (4.14) (4.15) of 5 more cases that can be found in previous DEA literature and the results are shown in Table 4.7 (where,  $\sigma$  represents standard deviation of the differences).

1. Evaluating regions in Serbia (30DMUs, 4 input 4 output variables) [21]
2. Location of solid waste system (22 DMUs, 5 input 3 output variables) [26]
3. Car selection problem (28 DMUs, 4 input 2 output variables) [20],[24]
4. Economic performance of Chinese cities (18 DMUs, 2 input 3 output variables) [42]
5. Location of hydro-electrical power station (6 DMUs, 4 input 2 output variables) [12], [20],[24]

**Table 4.7 Result of difference in 5 application examples**

Example	1	2	3	4	5
$D$	0.030	0.010	0.043	-0.010	0.183
$d_k$	0.090	0.022	0.104	-0.065	0.415
$\sigma$	0.021	0.007	0.023	0.033	0.132

Finally, we can see the followings in multiple input, multiple output cases from equation (4.12)

- 1) We cannot say in general that the difference between the two equations (4.7) and (4.8) is very small. However we can see that for a certain DMU  $k$ , when many values of  $\alpha_j$  are zero or very small then the difference of the results from two equations can be very small otherwise it will be rather large. The difference between two equations of DMU  $j$  is caused by the DMU  $j$ 's that favor far different weights and thus DMU  $j$  can not dominate each of DMU  $j'$  when using DMU  $j$ 's weight vectors.
- 2) Therefore, a certain CCR-efficient DMU  $j$ , which is maverick in the sample and cannot dominate many DMUs, will show rather large difference than CCR-efficient DMU  $j'$ , which is less maverick and can dominate many DMUs.
- 3) Among the 6 cases, the average differences in 5 cases are very small and only the result of case 5 (Location of hydro electrical power station) shows a relatively large difference. It is due to the fact that case 5 has very small number of DMUs compared to that of variables and the biplot of case 5 data shows that each DMU locates very sparse, which represents each DMU's favorite weight vector is far different.

#### 4.5 Conclusions

In this chapter, we showed that cross-efficiency evaluation in effect applies almost fixed weights in many of multiple-input, multiple-output cases, which is done as an extension of previous work [4] that focused on single-input, multiple-outputs case. We derived an equation, which shows the sources to make two results (real cross-efficiency score and that using fixed weights) be different and provided some explanations.

We also performed empirical study to confirm the results that cross-efficiency evaluation in effect applies almost fixed weights in many of multiple-input, multiple-output cases. Anderson et al [2] noted that the reasonability and acceptability of this model's fixed weights in single-input, multiple-output case depends on the judgment of the modeler. Actually, the 1<sup>st</sup> ranker in cross-efficiency evaluation can be considered as best performer with respect to the certain weight vectors, which are obtained by averaging each DMU's favorable weight vectors in the sample.



## **CHAPTER 5. THE CHARACTERISTICS OF CONE-RATIO WEIGHT RESTRICTIONS AND SOME EXPLANATIONS ON OTHER DEA ISSUES**

### **5.1 Introduction**

In chapter 3 and 4, we analyzed the characteristics of A-P model and cross-efficiency evaluation, which are used in case that we don't have any prior relative weights of inputs and outputs. One advantage of using these models is that the user need not identify prior relative weights of inputs and outputs. Unfortunately, the imputed input and output values of these models may be problematic when the user has certain value judgments that should be taken into account in the assessment and those values do not coincide with the imputed values actually applied in these models.

In chapter 5 and 6, we consider the characteristics of DEA ranking models with cone-ratio (hereafter, we call C/R) and Wong and Beasley weight restrictions, both of which take decision maker's opinion into account each of weight restrictions. The comprehensive range of weight restrictions that can be used to incorporate value judgments in DEA under constant returns to scale are well represented in [2], [38]. C/R DEA model was first initiated by (Charnes et al, 1990), in which assurance regions are defined by bounds on weights reflecting the relative importance of inputs or outputs.

In this chapter, we suggest two properties, (P1) and (P2) on the characteristics of C/R weight restriction (more specifically, assurance region type I by Thompson et al [39]). And using property (P1), we present graphical explanations of some other DEA issues such that 1) multiple solution problem 2) finding cross-efficiency multipliers 3) target points under C/R weight restrictions. By explaining some DEA issues graphically, which are proved

mathematically, we can get additional intuition. We think that these explanations using graph, even it is limited to the 2-dimensional case, can be useful which can provide intuitional knowledge for further analysis in some other issues in DEA.

The rest of this chapter is organized as follows. In section 5.2, we prove two properties (P1) and (P2) on the characteristics of C/R weight restrictions. Based on property (P1), we present graphical explanations on some other DEA issues. In section 5.3, after introducing theorems on classification and characterization of DMUs [10], we presented graphical explanation on multiple solution problems with example. Similarly in section 5.4 and section 5.5, we suggest graphical explanations on determining cross-efficiency multipliers and target point under C/R weight restrictions. Finally, conclusions are provided in section 5.5.

## **5.2 The characteristics of cone-ratio weight restrictions in DEA**

When we analyze the result of CCR model, most of previous DEA literatures have focused mainly on the value of each input and output multipliers rather than the ratio scales of them. In this chapter, we slightly changed our view in analyzing the result of CCR model like follows. That is, when we have the result of each input, output multipliers in CCR model, we consider that the corresponding cone-ratio weight vectors are assigned to each DMU  $j$  since they are most favorable.

For example in one input, two outputs case, if the CCR result for DMU  $j$  are  $\mu_1 = 0.111$ ,  $\mu_2 = 0.111$ , then we consider that DMU  $j$  took the output weight vector  $\mu_1 / \mu_2 = 1.00$  since it is most favorable rather than focusing each value itself. And we believe that this view enables us to have more clear interpretation on the characteristics of some DEA models when combined with the property (P1) below.



respectively. Since  $Q$  and  $Q'$ ,  $R$  and  $R'$ ,  $P$  and  $P'$  lie on each of iso-weight (preference) line, these points have the same weights respectively.

Therefore, the following relation (5.1) should hold that is also obvious by the property of right-angled triangle in  $\Delta OPP'$ .

$$\text{Efficiency score of (DMU P)} : \frac{OQ}{OP} = \frac{OQ'}{OP'}, \quad (5.1)$$

$$\text{Similarly, in case of (DMU B)} : \frac{OB}{OB} = \frac{OR'}{OR'}.$$

Now we define the vector  $OP'$  as the weight vector as follows.

**(D1)** The weight vector  $\bar{w}$  is a vector, which is perpendicular to the iso-weight lines (planes) of DMU  $j$  and passes through the origin.

Generally if there is a DMU  $j$ , which uses  $m$  inputs and the weights are given by C/R weight restrictions among inputs, then the iso-weight plane of DMU  $j$  can be expressed as  $v_1x_1 + \dots + v_mx_m = k$ . The equation of  $v_1x_1 + \dots + v_mx_m = k$  is the general form of the plane equation, which intersects each of  $m$  axes with the following points, i.e.

$$(k/v_1, 0, 0, \dots, 0), (0, k/v_2, 0, \dots, 0), \dots, (0, 0, 0, \dots, k/v_m), \text{ and } (v_1, \dots, v_m)$$

represents orthogonal (directional) vector to this plane that is passing through the origin.

Therefore we have the following definition on weight vector (D2).

**(D2)** The weight vector of DMU  $j$ , which uses  $m$  inputs and the weights are given by C/R weight ratios among inputs, can be represented such as input weight vector,

$$\bar{w}_i = (v_1, \dots, v_m) \text{ and the output weight vector, } \bar{w}_r = (\mu_1, \dots, \mu_r).$$

For simplicity, we assumed that the C/R weight for two inputs are the same for all DMUs in the case of Figure 5.1. Therefore we have all different iso-weight lines but parallel to each other to all DMUs and a unique weight vector. However in case that each DMU's C/R weight for each input is different, each DMU will have its own weight vector. After all, when the C/R weight restrictions (weight vectors) are applied to the general CCR model, all DMUs are projected to the weight vector along with the iso-weight lines (planes), and the efficiency score is measured by the following ratio (P1).

(P1) When the C/R weight restrictions (weight vectors) are applied to the CCR model, efficiency score of DMU  $j$  can be measured by the following ratio.

(single-input, multiple output or multiple-input, single output case)

*The efficiency score of DMU  $j$  =*

$$\frac{(\text{Norms of orthogonal projection of DMU } j \text{ to the weight vector})}{(\text{Norms of orthogonal projection of DMU } j^* \text{ to the weight vector})}$$

where, DMU  $j^*$  has the largest norm (output maximization case) or smallest norm (input minimization case) when projected to the weight vector.

(Proof)

When cone-ratio weight restrictions are applied in single-input, multiple-output situation under constant returns to scale, efficiency score of DMU  $j_o$  can be represented as

(5.2).

$$\begin{aligned} &\text{maximize} && \sum_{r=1}^j \mu_r y_{rj_o} \\ &\text{subject to} && v = 1 \end{aligned}$$

$$\begin{aligned}
\sum_{r=1}^s \mu_r y_{rj} - v &\leq 0, \quad j = 1 \cdots n \\
\frac{\mu_1}{\mu_2} &= k_1, \dots, \frac{\mu_1}{\mu_s} = k_{s-1}, \\
\mu_r &\geq 0, \quad \forall r
\end{aligned} \tag{5.2}$$

And (5.2) is equivalent to (5.3)

$$\begin{aligned}
&\text{maximize} \quad \sum_{r=1}^s \mu_r y_{rj_0} \\
&\text{subject to} \quad \sum_{r=1}^s \mu_r y_{rj} \leq 1, \quad j = 1 \cdots n \\
&\frac{\mu_1}{\mu_2} = k_1, \dots, \frac{\mu_1}{\mu_s} = k_{s-1}, \\
&\mu_r \geq 0, \quad \forall r
\end{aligned} \tag{5.3}$$

Applied cone-ratio weight ratios can be represented as the weight vector (5.4) in output multiplier space and all DMU's iso-preference planes are orthogonal to this weight vector.

$$\bar{w}_r = (\mu_1, \dots, \mu_r) \tag{5.4}$$

$$\text{Therefore, efficiency score of DMU } j_0 = \frac{\sum_{r=1}^s \mu_r y_{rj_0}}{\left( \sum_{r=1}^s \mu_r y_{rj} \right)_{\max}} = \frac{\sum_{r=1}^s \mu_r y_{rj_0}}{1}$$

$$= \frac{(\text{Norms of orthogonal projection of DMU } j \text{ to the weight vector})}{(\text{Norms of orthogonal projection of DMU } j^* \text{ to the weight vector})}$$

However in multiple-input, multiple-output case, we cannot represent the efficiency of DMU  $j$  using 2-dimensional figure and thus we have to say that

*The efficiency score of DMU  $j$  =*

$$\frac{\text{virtual output of DMU } j / \text{virtual input of DMU } j}{\text{virtual output of DMU } j^* / \text{virtual input of DMU } j^*}$$

where, DMU  $j^*$  has the largest efficiency score with the same input and output weight vectors with DMU  $j$  respectively.

Generally in DEA, it is admitted that C/R weight restrictions represent the decision maker's decision of relative importance on each variable. Also this relative importance has been explained as the marginal rates of substitution between inputs or between outputs [8], [38]. However, the fact that C/R weight restrictions in DEA in multiple-input, multiple-output case don't imply the relation of perfect substitution among inputs (or outputs) has not been emphasized in previous DEA literature.

**(P2)** While the C/R weight ratios among inputs (or outputs) in one-input, multiple-output case imply the relations of perfect substitution, it is not exactly true in multiple-input, multiple-output case.

(P2) can be explained by the simple counter example of two-input, two-output case.

When output weight ratio is given as  $\mu_1/\mu_2 = k$ , a certain CCR inefficient DMU may not get the same efficiency score while increasing each output according to the weight ratio. That is, the following two cases may result in different efficiency score

(Case 1) :  $(x_1, x_2), (y_1 + \Delta y_1, y_2)$ , (Case 2) :  $(x_1, x_2), (y_1, y_2 + k\Delta y_1)$ .

We explain above properties using the following example 5.1. Table 5.1 shows the data set (left table) and the CCR results of each DMU (right table). The final column in Table 5.1 shows the ratio of output multipliers of each DMU, in which we can see that only DMU 3

chose the ratio of output multipliers  $\mu_1 / \mu_2$  as 2.00 and all the other five DMUs chose the ratio of 1.00.

**Table 5.1 Data and CCR results of example 5.1**

Data				CCR Multipliers - Input Oriented				
DMU	$x$	$y_1$	$y_2$	CCR	$v_1$	$\mu_1$	$\mu_2$	$\mu_1 / \mu_2$
1	1	1.0	8.0	1	1	0.1111	0.1111	1.00
2	1	4.0	5.0	1	1	0.1111	0.1111	1.00
3	1	6.0	1.0	1	1	0.1538	0.0769	2.00
4	1	1.0	7.0	0.8889	1	0.1111	0.1111	1.00
5	1	1.5	7.0	0.9444	1	0.1111	0.1111	1.00
6	1	1.5	6.5	0.8889	1	0.1111	0.1111	1.00

Figure 5.2 is drawn with overlapping two planes, one is input-output variable plane  $(y_1/x, y_2/x) = (y_1, y_2)$  and the other is corresponding multiplier plane  $(\mu_1/v, \mu_2/v) = (\mu_1, \mu_2)$  in this case. And we showed the weight vectors  $\mu_1 = \mu_2$ ,  $\mu_1 = 2\mu_2$  in this overlapped plane.

First, when we assume that the decision maker's weight (preference) for two outputs are given to equal and apply this preference as C/R weight restriction  $\mu_1 / \mu_2 = 1.00$ , then the efficiency score of DMU  $j$  can be measured by the suggested property. That is, 1)  $\overline{1E2A}$ ,  $\overline{5B}$ ,  $\overline{46C}$ ,  $\overline{3D}$  represent the iso-weight (preference) lines for each DMU, where A, B, C and D represent the projection points of DMUs 1, 5, 4 (6) and 3 respectively 2) Therefore, the efficiency score of each DMU can be measured as follows.  $\frac{OA}{OA} = 1$  (DMUs 1 and 2),  $\frac{OD}{OA} = 0.777$  (DMU3),  $\frac{OC}{OA} = 0.888$  (DMUs 4 and 6)  $\frac{OB}{OA} = 0.944$  (DMU 5).





$\mu_1 = 2\mu_2$ . Therefore if the DMU's weight (preference) ratio is  $\mu_1 = \mu_2$ , the overall efficiency score of DMU 3 is less than technical efficiency score of 1.

To confirm the above approach, we showed the calculation results on the each DMU's projection points and ratio of the norms of projection (i.e. efficiency score) using basic theory of linear algebra in appendix A-8.

Until now, we showed the way of measuring efficiency when we have C/R weight restrictions in DEA. Since the C/R weight restriction allows flexible substitution, it can have possible drawback when the decision maker's weight (preference) on input (output) variables doesn't imply (or allow) the substitution among inputs or outputs at all or allow the substitution only in certain ranges, i.e. not allow the relation of perfect substitution.

That is, there can be a case that even though (decision maker's) revealed relative importance of the two outputs is  $(\mu_1 : \mu_2) = (1 : 1)$ , which means the marginal rate of substitution between two outputs is  $-1$ , i.e. a 1 unit increase of output 1 would be compensated for by a 1 unit decrease of output 2, but it doesn't mean that it is also acceptable (no difference) in the following case 2 or case 3. 1) Case 1 : (output 1: output 2) = (3 units: 1 unit) and (1 unit : 3 units) 2) Case 2 : (output 1: output 2) = (100 units: 1 unit) and (1 unit : 100 units) 3) Case 3 : (output 1: output 2) = (1000 units: 1 unit) and (1 unit : 1000 units).

### 5.3 Graphical explanations on the multiple solution problems in DEA

#### 5.3.1 Classification and characterization of DMUs

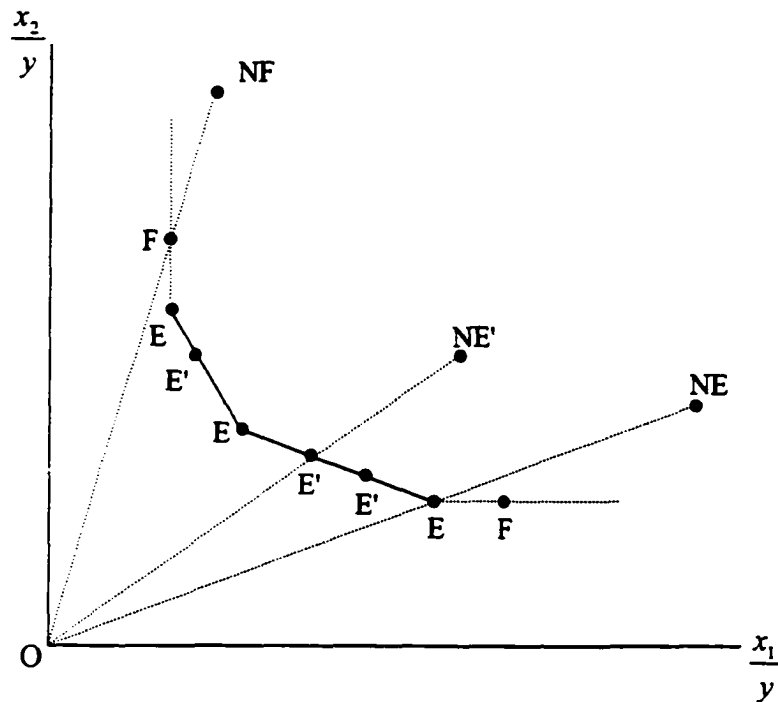


Figure 5.3 Classification of DMU efficiencies

Charnes et al [9],[10] suggested on the classification and characterization of DMUs into 6 classes shown in Figure 5.3.

1. The set of all DMUs is partitioned into 6 classes :  $E, E', F, NE, NE', NF$  .
2. DMUs  $E, E', F$  are scale efficient but only DMUs  $E, E'$  are Pareto-Koopmans efficient.
3. DMU  $E$  is efficient and is characterized by the property that their sets of optimizing multipliers are all of the maximal dimension  $s + m$  . This means that all of DMU  $E$  has the multiple solutions in CCR model and the sets of each multiplier are linearly independent with maximal dimension  $s + m$  .

4. DMU  $E'$  are also efficient and have at least one optimizing multiplier with all component positive, however they differ from  $E$  in that their set of optimizing multipliers have dimension less than  $s + m$ .
5. Both  $E$  and  $E'$  are on the Pareto-Koopmans efficiency frontier of  $K$ .
6. DMU  $F$  is also on the frontier  $K$  but is associated with DMU that is not efficient. This has no optimizing multiplier in which all components are strictly positive.

The definition 1 of CCR-Efficiency in chapter 2 can be restated in relation to each DMU as follows.

- (1) DMUs  $E \cup E'$  are CCR efficient and there exists at least one optimal  $(\nu^*, \mu^*)$ , with  $\nu^* > 0$  and  $\mu^* > 0$ .

The definition 2 of CCR-Efficiency in chapter 2 can be restated in relation to each DMU as follows.

- (2)  $E \cup E'$ :  $\theta = 1$  and  $\sum s_r^+ + \sum s_i^- = 0$
- $F$ :  $\theta = 1$  and  $\sum s_r^+ + \sum s_i^- > 0$
- $NE \cup NE'$ :  $\theta < 1$  and  $\sum s_r^+ + \sum s_i^- = 0$
- $NF$ :  $\theta < 1$  and  $\sum s_r^+ + \sum s_i^- > 0$

### 5.3.2 Explanations on the multiple solution problems in DEA

Charnes et al's papers [9], [10] on the structure for classifying and characterizing efficiency and inefficiency in DEA have been a strong basis for further analysis in DEA. From the above explanation of each DMU's characterization, we could see that DMUs  $E$  and  $E'$  may have multiple solutions based on the explanation that their sets of optimizing

multipliers for DMUs  $E$  and  $E'$  have all of the maximal dimension  $s + m$  and have less than  $s + m$  respectively.

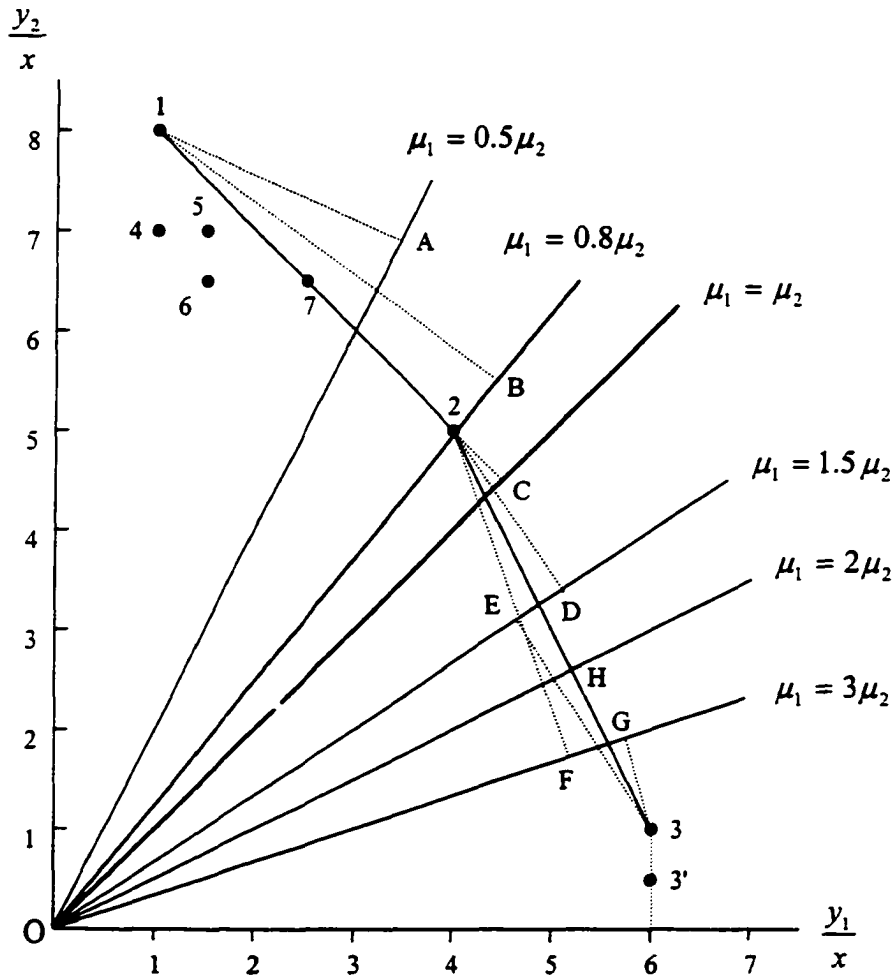
While they [9], [10] suggested and mathematically proved the theorems on the multiple solution problems in DEA, this problem has been explained more specifically (example of two input, one output case) using the dual form of CCR model in several papers [27], [36], [43] that certain DMUs can have multiple  $\lambda$  values in their optimal solutions. It seems due to the fact that dual form of CCR model has far less constraints and it makes easier to perform simplex tableau. But we cannot get any graphical intuition on the possible multiple solutions using the dual form of CCR model, which makes it more difficult to understand.

In this section, we suggest the explanation of multiple solution problems using the primal form of CCR model based on the property (P1).

We use the example 5.1 data (Table 5.1) and Figure 5.4 shows the projection of DMU 1, 2 and 3 to various weight vectors.

- (1) At first, DMU 1 has the same projection point at C (4.5, 4.5) with DMU 2 when projected to the weight vector  $\mu_1 = \mu_2$  which results in efficiency score 1 of both DMUs 1 and 2. However if projected to any weight vector in relation of  $\mu_1 < \mu_2$  (we showed in Figure 5.4 the two cases of  $\mu_1 = 0.5\mu_2$  and  $\mu_1 = 0.8\mu_2$ ), DMU 1 is the only one which has the efficiency score of 1, i.e. DMU 1 dominates all other DMUs. This implies that DMU 1 has multiple optimal solutions.
- (2) DMU 2 has the same projection point at H (5.2, 2.6) with DMU 3 when projected to the weight vector  $\mu_1 = 2\mu_2$  which results in efficiency score 1 of both DMU 2 and 3. However if projected to any weight vectors  $\mu_2 < \mu_1 < 2\mu_2$  (we showed in Figure 5.4

the one case of  $\mu_1 = 1.5\mu_2$ ), DMU 2 is the only one which has the efficiency score of 1, i.e. DMU 2 dominates all other DMUs. This also implies that DMU 2 also has multiple optimal solutions.



**Figure 5.4 The range of optimal multipliers for DMUs 1 and 2**

- (3) For DMU 3, if projected to any weight vectors  $\mu_1 > 2\mu_2$  (we showed in Figure 5.4 the one case of  $\mu_1 = 3\mu_2$ ), DMU 3 is the only one which has the efficiency score of 1, i.e. DMU 3 dominates all other DMUs. This also implies that DMU 3 has multiple optimal solutions.

- (4) If we assume that there is another DMU 3' which corresponds to  $F$  in DMU classification of Charnes et al [10], the only possible weight vector with which the efficiency score of DMU 3' is 1 is  $\mu_2 = 0$ . With any weight vector, DMU 3' cannot be the only one which has the efficiency score of 1. But only when it takes the weight vector  $\mu_2 = 0$ , the efficiency score is 1 equal to DMU 3. This also coincides with the definition 1 of CCR- efficiency : DMUs  $E \cup E'$  are CCR efficient and there exists at least one optimal  $(\nu^*, \mu^*)$ , with  $\nu^* > 0$  and  $\mu^* > 0$ .
- (5) DMUs 1, 2 and 3 are belong to  $E$  in DMU classification of Charnes [10] and all of corresponding optimal multipliers are linearly independent. Therefore their sets of optimizing multipliers have the maximal dimension  $s + m = 3$ .
- (6) If we assume that there is another DMU 7 shown in Fig 5.4, which belongs to  $E'$  in DMU classification of Charnes et al [10], it can have efficiency score 1 only when projected to the weight vector  $\mu_1 = \mu_2$ . Therefore DMU 7 has the unique solution with relation of  $\mu_1 = \mu_2$ , and the dimension is  $2 < s + m = 3$ .
- (7) CCR- inefficient DMUs, i.e. DMU 4, 5 and 6 all belong to  $NE'$  in DMU classifications. They will have the unique solution like DMU 7 ( $E'$ ) since they can have their maximum efficiency score only when projected to the weight vector  $\mu_1 = \mu_2$ . Therefore the dimension is  $2 < s + m = 3$ .

After all DMUs 1, 2 and 3 have multiple optimal solutions in their set of multipliers with which these DMUs can be evaluated as technically efficient (i.e. efficiency score is 1).

Therefore when we apply the CCR model to each DMU with any of these optimal ratios of multipliers with an added constraint, the efficiency score will not be changed.

For example, DMU 1 is dominant as long as  $\mu_1 < \mu_2$ , we randomly chose an added constraint  $\mu_1 / \mu_2 = 0.30$ . Also DMU 2 is dominant as long as  $\mu_2 < \mu_1 < 2\mu_2$ , we randomly chose an added constraint  $\mu_1 / \mu_2 = 1.50$ . Similarly for DMU 3, we chose  $\mu_1 / \mu_2 = 3.00$ . Even though multipliers are changed, CCR-efficiency score is the same as before. For DMUs 4, 5 and 6 any other ratios of multiplier weights  $\mu_1 / \mu_2 = 1$ , cannot make higher CCR-efficiency score than before.

Linear programming algorithms typically are terminated when a single optimal solution is obtained, without fully characterizing the set of optimal solutions. Therefore, just from CCR result, we often cannot get sufficient information about the applicable weights on each DMU's input / output variable to obtain maximum technical efficiency.

## **5.4 Graphical explanations on the other issues in DEA**

### **5.4.1 The multipliers of cross-efficiency evaluation**

In this section we will examine how the cross-efficiency evaluation determines its optimal multipliers for each DMU using the example 5.1 data.

In cross-efficiency evaluation, the objective function is to minimize the summation of all other DMU's virtual output with the following constraints. 1) The summation of all other DMU's virtual input is equal to 1 2) The efficiency score of all other DMUs cannot exceed 1 while keeping the CCR-efficiency score of DMU being evaluated.



The above formulation is often explained as follows [4]. “ This attempts to mitigate multiple solutions and is a process by which for each DMU, given its initial CCR-efficiency score, one of the available weighting schemes is selected to itself and others ”

However we believe that even though above expression is conceptually well known, the way how cross-efficiency evaluation selects one optimal solution among multiple solutions has not been specifically shown in the previous research.

Table 5.2 shows the cross-efficiency multipliers (left table) and scores with cross-efficiency matrix (right table) of example 5.1 data. The ratios of output multipliers are changed only to CCR-efficient DMUs 1, 2 and 3 and for the case of DMUs 4, 5 and 6 the ratios of output multipliers are not changed. DMU 2 got the highest cross-efficiency score of 0.882.

**Table 5.2 Cross-efficiency results of example 5.1**

Cross-efficiency Multipliers					Cross-efficiency Matrix						
DMU	$v_1$	$\mu_1$	$\mu_2$	$\mu_1 / \mu_2$	DMU	1	2	3	4	5	6
1	0.2	0	0.025		1	1.000	0.625	0.125	0.875	0.875	0.813
2	0.2	0.0308	0.0154	2.00	2	0.769	1.000	1.000	0.692	0.769	0.731
3	0.2	0.0333	0		3	0.167	0.667	1.000	0.167	0.250	0.250
4	0.2	0.0222	0.0222	1.00	4	1.000	1.000	0.778	0.889	0.944	0.889
5	0.2	0.0222	0.0222	1.00	5	1.000	1.000	0.778	0.889	0.944	0.889
6	0.2	0.0222	0.0222	1.00	6	1.000	1.000	0.778	0.889	0.944	0.889
mean	0.2	0.0218	0.0178	1.22	mean	0.823	0.882	0.743	0.733	0.788	0.743

Figure 5.5 shows the projections of each DMU taken by cross-efficiency to each weight vector. DMU 1 preferred the weight vector  $\mu_1 = 0$  and DMU 3 preferred  $\mu_2 = 0$ . It is clear in Figure 5.5 that with this respective projection, DMU 1 and 3 can suppress the sum of all the other DMU's efficiency score as much as possible.

In the single-input, multiple-outputs case with all inputs are unified to 1, to minimize the sum of all the other DMU's virtual output is equivalent to minimizing the sum of all the other DMU's efficiency scores. Here DMU 2 changed its weight vector from  $\mu_1 = \mu_2$  (CCR) to  $\mu_1 = 2\mu_2$  (cross-efficiency evaluation). The reason is that it is the best weight among CCR multiple solutions, which can suppress the sum of all the other DMU's efficiency score. That is, when we compare the value of sum of all the other DMU's efficiency score for DMU 2, when projected to  $\mu_1 = 2\mu_2$ , it will be smaller than projected to  $\mu_1 = \mu_2$ .

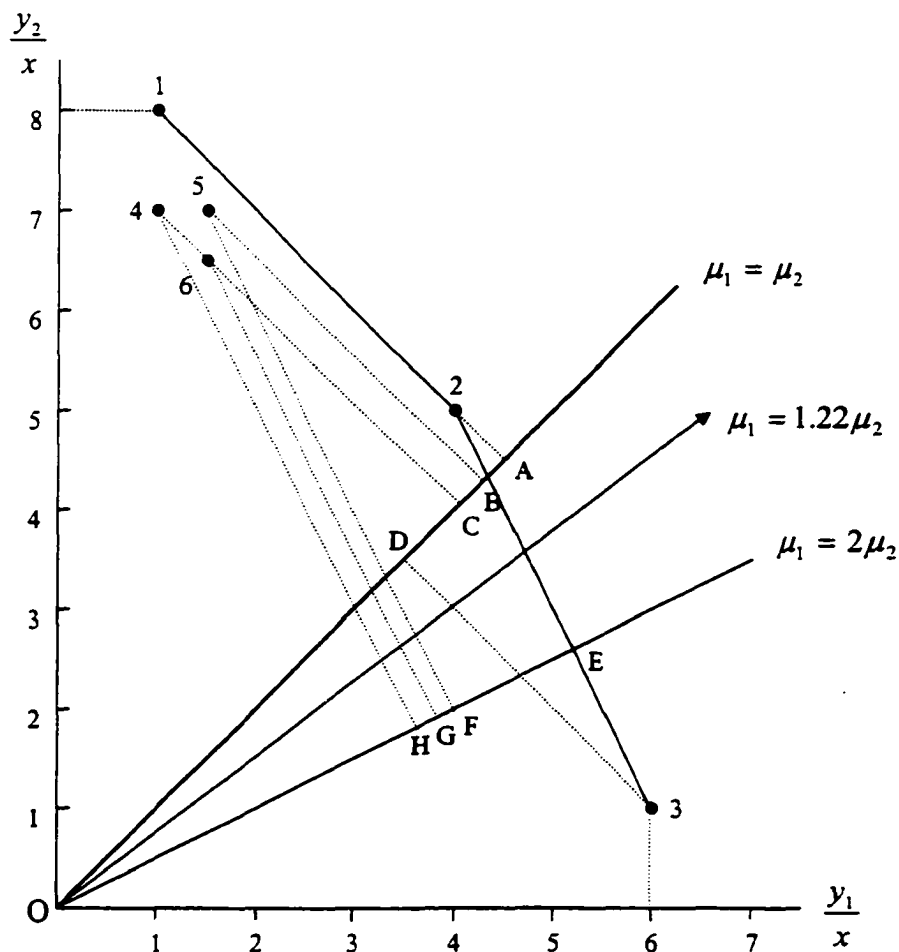


Figure 5.5 The multipliers of cross-efficiency evaluation

$$\begin{aligned} & \left(\frac{OA}{OA}\right)_1 + \left(\frac{OD}{OA}\right)_3 + \left(\frac{OC}{OA}\right)_4 + \left(\frac{OB}{OA}\right)_5 + \left(\frac{OC}{OA}\right)_6 \\ & = 1 + 0.777 + 0.888 + 0.944 + 0.888 = 4.497 \text{ for } \mu_1 = \mu_2 \end{aligned}$$

$$\begin{aligned} & \left(\frac{OF}{OE}\right)_1 + \left(\frac{OE}{OE}\right)_3 + \left(\frac{OH}{OE}\right)_4 + \left(\frac{OF}{OE}\right)_5 + \left(\frac{OG}{OE}\right)_6 \\ & = 1 + 1 + 0.692 + 0.769 + 0.731 = 4.192 \text{ for } \mu_1 = 2\mu_2. \end{aligned}$$

In chapter 4, we explained that cross-efficiency applies implicitly fixed weights, which is exactly equal to the column mean of each multiplier, to each and every DMU in a single-input, multiple-output under constant returns to scale. In this example, the ratio of these weights  $\mu_1 = 1.22\mu_2$  is shown in the final row in Table 5.2 and also displayed in Fig 5.5. However when all DMUs are projected to this weight vector, the efficiency scores of each DMU will not be same as those of cross-efficiency score. Because this kind of cone-ratio model always makes at least one efficient DMU, i.e. efficiency score is 1 (in this case DMU 2). But the scores are changed proportionally, which still results in the same ranking.

#### 5.4.2 Target points under cone-ratio weight restriction

Graphical explanation also can be provided on the target points under the cone-ratio weight restriction. Recently, Thanassoulis [38] suggested and proved mathematically the following statements on the targets under cone ratio type weight restriction.

- (1) “ If we expand the output levels of DMU  $j_o$  by the factor  $Z^*$  without increase any of its input level, DMU  $j_o$  will be rendered 100% efficient, but the resulting input-output levels may not lie within the production possibility set ”, where  $Z^*$  is the

efficiency score of DMU  $j_o$  under the cone ratio type weight restriction.

- (2) “ Thus, when at least one weight restriction is binding in a DEA model we cannot use the DEA efficiency rating as a simple scaling constant to estimate expansions of output levels or contractions of input levels which are feasible in principle under efficient operation”.

Let us assume that DMU 7, which produces 2 outputs (5, 2) using 1 input (1), is added to the example 5.1 data.

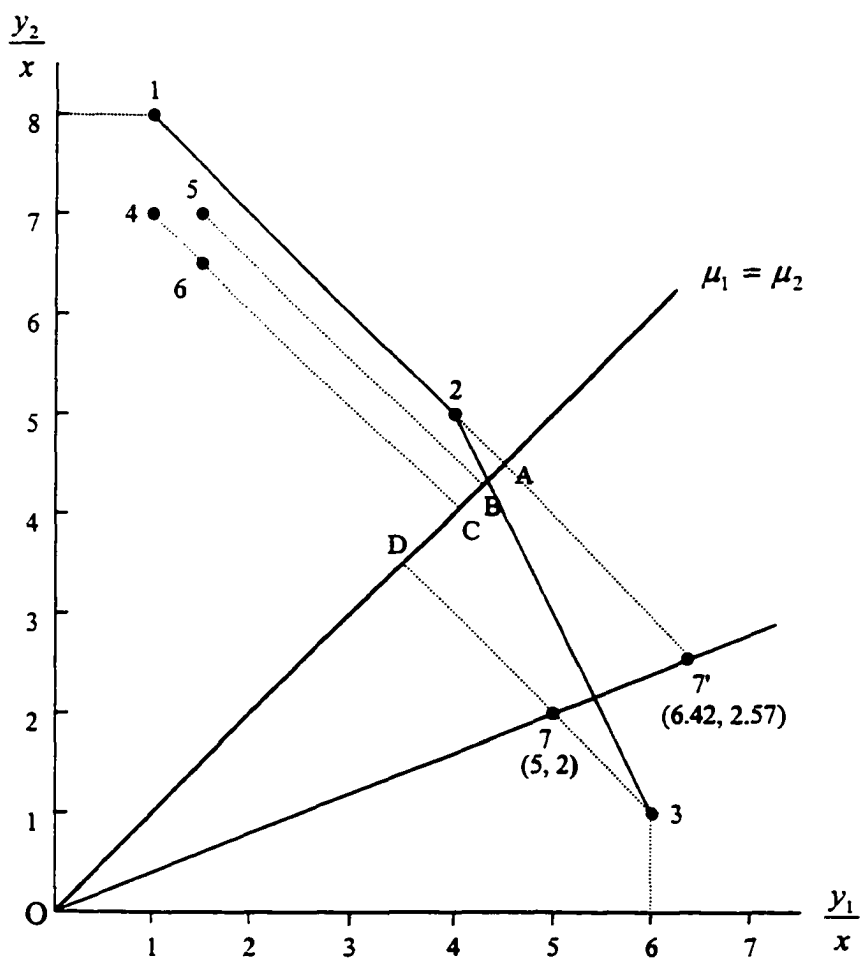


Figure 5.6 Illustration of the target point of DMU 7

The efficiency score of DMU 7 with restriction of  $\mu_1 = \mu_2$  is 0.777 and is also equal to that

of DMU 3. Then above statements can be easily confirmed from Figure 5.6.

(1) Because efficiency score of DMU 7 is 0.777, if we calculate the 100% efficient point

(7') by the same way as CCR model without restriction, it would be  $(\frac{1}{\theta} \times 5, \frac{1}{\theta} \times 2) =$

(6.42, 2.57). Clearly this is outside the production possibility set.

(2) It is also clear that the projection of DMUs 1, 2 and 7' to the weight vector  $\mu_1 = \mu_2$  is

equal to OA. Therefore, if we have to find the 100% efficiency within the production

possibility set, efficient target of DMU 7 can only be (4, 5), which is the output

levels of DMU 2.

## 5.5 Conclusions

In this chapter, we proved property (P1) that shows the way to measure efficiency when cone-ratio weight restrictions are applied under constant returns to scale with one- input, multiple-outputs (or multiple-inputs, one-output) in DEA. Based on this property, we proposed some graphical explanations of other DEA issues, 1) multiple solution problem 2) multipliers of cross-efficiency evaluation 3) target points under cone-ratio weight restrictions using one-input, two-output case in DEA. We believe that graphical explanation can be useful, even if it is limited in 2-dimensional case, which provides simple but intuitional knowledge for further analysis in many cases.

## CHAPTER 6. THE COMPARISONS BETWEEN CONE-RATIO AND WONG AND BEASLEY WEIGHT RESTRICTIONS IN DEA

### 6.1 Introduction

Rather than restricting the actual weights, Wong and Beasley [41] suggested a method that restricts virtual inputs or outputs. (Here, we used the expression of ‘restricting actual weights’ to follow the classification of Allen [2] and Thanassoulis [38]). The method of restricting the actual weights indicates the models presented in section 2.7.2. (That is, assurance region type I, II and the absolute weight restrictions)

They proposed that

(1) The proportion of output  $r$  devoted to the total outputs for DMU  $j$  can be represented

as  $\frac{\mu_r y_{rj}}{\sum_{r=1}^s \mu_r y_{rj}}$  and it means the ‘importance’ attached to output measure  $r$  by DMU  $j$ .

(2) It is because the larger this value, the more DMU  $j$  depends on output measure  $r$  in determining its efficiency.

Based on the decision maker’s *value judgments*, we can set the lower and upper limits for the importance of output  $r$  in DMU  $j$  and this can be expressed shown in (6.1).

$$\alpha_r \leq \frac{\mu_r y_{rj}}{\sum_{r=1}^s \mu_r y_{rj}} \leq \beta_r \quad (0 \leq \alpha \leq \beta \leq 1) \quad (6.1)$$

Shang et al [29] showed an application using this Wong and Beasley weight restrictions (hereafter we call W/B weight restrictions) with weights obtained by AHP (Analytic Hierarchy Process). However they didn’t mention why W/B weight restrictions are most appropriate in that case (weights obtained by AHP) and the differences between using cone-

ratio weight restrictions.

Also on the W/B weight restrictions Allen et al [2] indicated that

- (1) Even though restrictions on virtual input or output weights have received relatively little attention in DEA literature, more research is necessary to explore the *pros* and *cons* of setting restrictions on the virtual inputs and outputs.
- (2) Heretofore, there has been no attempt to compare methods for setting restrictions on the actual DEA weights with those restricting virtual inputs and / or outputs.

Here in this study we limit our focus on the cone-ratio type weight restrictions (more specifically type  $r_2$  and  $r_4$  in model (2.8) in section 2-7-2) among the actual weight restriction methods to compare with W/B weight restrictions. Therefore hereafter we consider the actual weight restrictions as cone-ratio (C/R) weight restrictions.

In this chapter, we compared the characteristics between two restriction methods C/R and W/B in DEA. After discussing theoretical difference between two weight restriction methods, we compared the characteristics using simple example of one-input, two-output case and also compared empirically using applications of multiple-input, multiple-output case.

The rest of this chapter is organized as follows.

In section 6.2, the characteristics of W/B weight restriction are discussed theoretically based on single-input, multiple-output situation with example. In this section, we showed that under W/B weight restriction, each DMU takes all different weight vectors and some DMUs may have limiting efficiency score. In section 6.3, we introduced AHP to get the weights in DEA based on previous research by Shang et al [29] and compared the results of C/R and W/B weight restriction using given single data. The ranking results are appeared very similar in

that case. To see the practical difference between two restriction methods, in section 6.4, we performed empirical study using two application problem data, each of 6 random weight cases. The ranking results of all 12 cases are still appeared very similar. Also we showed that the ranking result may be far different when a certain DMU has limiting efficiency score under W/B weight restriction. Finally, conclusions are provided in section 6.5.

## 6.2 The characteristics of W/B weight restrictions

Based on the decision maker's value judgments, W/B weight restriction sets the lower and upper limits for the importance of output  $r$  of DMU  $j$  like equation (6.1). When we think the case of one input, two outputs  $(y_1, y_2)$  with the following W/B weight restriction (6.2), each DMU takes weight vectors by the relationship of (6.3).

$$\frac{\mu_1 y_1}{\mu_1 y_1 + \mu_2 y_2} = \alpha, \quad \frac{\mu_2 y_2}{\mu_1 y_1 + \mu_2 y_2} = \beta \quad (6.2)$$

$$\text{then } \mu_1 y_1 : \mu_2 y_2 = \alpha : \beta \quad \Rightarrow \quad \alpha \mu_2 y_2 = \beta \mu_1 y_1$$

$$\text{therefore } \frac{\mu_1}{\mu_2} = \frac{\alpha y_2}{\beta y_1} \quad (6.3)$$

Also it is true that in the multiple-input, multiple-output case, we can get all pairwise weight vectors among inputs and outputs like the same manner above. Therefore W/B weight restriction can be viewed another type of C/R weight restriction. But the main difference in W/B weight restriction is each DMU takes the different weight vectors under the same given criteria (given preferences).

To explain the characteristics of W/B weight restriction, we use the example 5.1 data again and also assume the equal W/B weights (importance) on two outputs such that  $(\alpha : \beta = 0.5 : 0.5)$  in (6.2). Table 6.1 shows the results comparison between C/R (with



restriction of  $\mu_1 = \mu_2$ ) and W/B weight restriction ( $\alpha : \beta = 0.5 : 0.5$ ) in (6.2). The columns under C/R and W/B represent the efficiency scores when each weight restriction is applied. Only DMU 2 has the same efficiency score and all the other DMUs have much lower scores when applied W/B restriction.

The way for measuring efficiency score can be explained by using weight vectors in Figure 6.1 in a similar way as in C/R weight restriction. For example, DMU 1 takes the weight vector  $\mu_1 = 8\mu_2$ , which can be seen from the last column in Table 6.1 and the efficiency score is measured by  $\frac{OA}{OC}$ . The coordinates of A and C are A(1.9692, 0.2462), C(6.0307, 0.7538).

$$\text{Therefore } \frac{OA}{OC} = \frac{\sqrt{1.9692^2 + 0.24615^2}}{\sqrt{6.030769^2 + 0.753846^2}} = \frac{1.9845}{6.0777} = 0.3265, \text{ which is exactly same with}$$

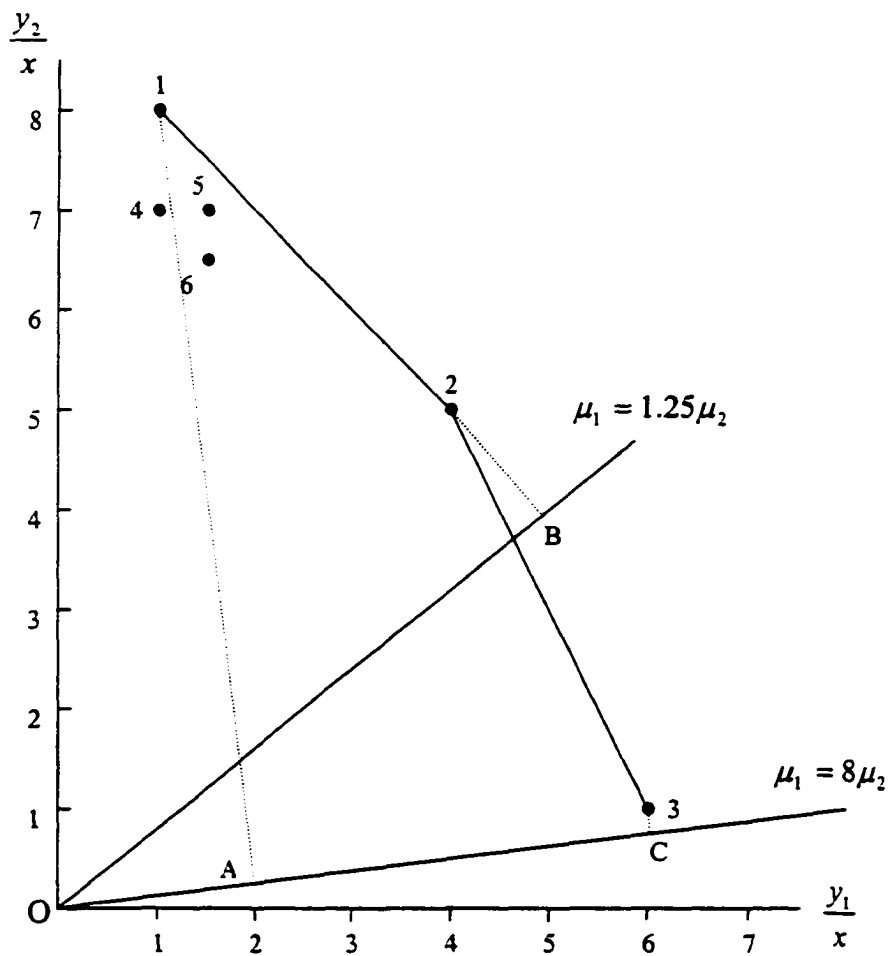
the results in Table 6.1. On the other hand, DMU 2 takes the weight vector  $\mu_1 = 1.25\mu_2$  and the efficiency score is  $OB/OB = 1$ , where the coordinate of B is B(4.878, 3.9024).

Here we indicate the followings on W/B weight restriction in above example.

- 1) When the C/R weight restriction is applied, all DMUs take the same weight vector  $\mu_1 = \mu_2$  and the efficiency score is measured by the ratio of norms of projections to the same weight vector. However when W/B weight restriction is applied, each DMU takes a different weight vector and the efficiency score is measured by the ratio of projection norms on that vector.
- 2) W/B weight restriction makes pretty low efficiency scores for some DMUs. In this case, all DMUs except DMU 2 got much lower score than those under C/R weight restriction.

**Table 6.1 Results comparison between C/R and W/B weight restriction**

DMU	C/R	Multipliers (C/R)				W/B	Multipliers (W/B)			
		$v_1$	$\mu_1$	$\mu_2$	$\mu_1 / \mu_2$		$v_1$	$\mu_1$	$\mu_2$	$\mu_1 / \mu_2$
1	1	1.0	0.111	0.111	1.00	0.327	1.0	0.163	0.020	8.00
2	1	1.0	0.111	0.111	1.00	1	1.0	0.125	0.100	1.25
3	0.778	1.0	0.111	0.111	1.00	0.245	1.0	0.020	0.122	0.17
4	0.889	1.0	0.111	0.111	1.00	0.326	1.0	0.163	0.023	7.00
5	0.944	1.0	0.111	0.111	1.00	0.483	1.0	0.161	0.034	4.67
6	0.889	1.0	0.111	0.111	1.00	0.481	1.0	0.160	0.037	4.33

**Figure 6.1 Illustration of W/B weight restriction**

Actually, as long as there is any DMU which keeps its production of  $y_1 = 1$ , no matter how

many units of  $y_2$  are produced it can not beat DMU 5 in its efficiency score (0.483) under W/B weight restriction ( $\alpha : \beta = 0.5 : 0.5$ ) in (6.2).

For example, if we imagine DMU 7, which produces two outputs  $(y_1, y_2) = (1, \infty)$ , efficiency score under W/B weight restriction will be 0.333, which is still lower than that of DMU 5.

- 3) Another issue is that we can't even use or define W/B restrictions if any of inputs or outputs is zero.
- 4) When we use W/B weight restriction we need to add the constraints (6.1). The number of constraints is  $2 \times (m + s)$  corresponding to upper and lower limits of each input and output variable. Therefore we need to change the constraints  $\{2 \times (m + s)\} \times j$  times to calculate all DMU's efficiency scores.
- 5) While C/R weight restriction allows flexible substitution among inputs or outputs, W/B weight restriction would not allow sufficient substitution (inflexible). Therefore, in order to reflect the decision maker's preferences more precisely, we have to know the decision maker's preferences more clearly and then decide which restriction method is more appropriate. Another example to show the difference between C/R and W/B weight restriction is suggested in appendix A.9.

### **6.3 Using AHP (Analytic Hierarchy Process) to get the weights in DEA**

AHP [25] is designed for subjective evaluation of a set of alternatives based on multiple criteria, organized in a hierarchical structure. The purpose of AHP is to provide a vector of weights expressing the relative importance of several elements (units). AHP can be used to

reflect judgments on feelings, ideas, and emotions. The output of the AHP is a prioritized ranking, indicating the overall preference for each decision alternative.

Shang et al [29] introduced an application (Performance evaluation of FMS) using W/B weight restrictions with weights obtained by AHP. In [29], the AHP weights (upper and lower bound) of each input and output was decided by pairwise comparison of relative importance such that 1) output improvement from each input, 2) input needed to improve each output.

They focused on using W/B weight restriction based on the following property (6.6) of AHP without mentioning the availability of C/R weight restriction. However, just from the decision criteria 'pairwise comparison of relative importance' and the property (6.6), we cannot clearly say that W/B weight restriction is more appropriate than C/R weight restriction, since C/R weight restriction can also satisfy both of above.

The purpose of this section does not lie on the detailed description of AHP but to show an alternative way of C/R weight restriction and compare the results using weights obtained by AHP. Therefore we have to make clear that a detailed description of AHP theory and FMS selection are out of scope of thesis and much of explanations on AHP and corresponding data of FMS evaluation are followed to Shang et al [29].

For illustration purpose, we begin with brief explanation of AHP procedure

*Step 1.* From a decision maker's pairwise comparison between the  $i^{\text{th}}$  and  $j^{\text{th}}$  inputs, the quantified judgment of the pair is recorded as numerical entries  $a_{ij}$  in a matrix (A)

*Step 2.* The AHP changes the information of A into a weight vector

$W = (w_1, w_2, \dots, w_m)^T$  representing the importance or contribution of each input

in relation to the improvement of each output. The  $i^{\text{th}}$  component of  $w_i$  of  $W$  is determined in a manner that it satisfies  $w_i/w_j = a_{ij}$  in  $A$ . That is

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1m} \\ a_{21} & 1 & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1m} \\ 1/a_{12} & 1 & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1m} & 1/a_{2m} & \cdots & a_{mm} \end{pmatrix} = \begin{pmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_m} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_m} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & \cdots & \frac{w_3}{w_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_m}{w_1} & \frac{w_m}{w_2} & \cdots & \frac{w_m}{w_m} \end{pmatrix}$$

and with the two unique properties (6.4) and (6.5)

$$AW = mW \quad (6.4)$$

$$\sum_{i=1}^m w_i = 1 \quad (6.5)$$

Shang et al [29] consider that  $w_i$  can be substituted for  $w_i \cong v_i x_i / \sum_{i=1}^m v_i x_i$ , because of the following properties (6.6).

$$(a) \sum_{i=1}^m w_i = \sum_{i=1}^m \left( \frac{v_i x_i}{\sum_{i=1}^m v_i x_i} \right) = 1, \quad (b) a_{ij} = \frac{w_i}{w_j} \cong \frac{v_i x_i / \sum_{i=1}^m v_i x_i}{v_j x_j / \sum_{j=1}^m v_j x_j} = \frac{v_i x_i}{v_j x_j} \quad (6.6)$$

However more strictly speaking, each  $w_i$  obtained from AHP can be substituted for W/B weight restriction when the weights in AHP are obtained by decision makers based on the relation  $w_i \cong v_i x_i / \sum_{i=1}^m v_i x_i$ . On the other hand, if the weights in AHP are obtained by decision makers based on the different relation, i.e.  $w_i \cong v_i / \sum_{i=1}^m v_i$ , we have to use C/R weight restriction instead of W/B weight restriction.

Therefore, we consider the case that the weights in AHP are obtained based on the relation  $w_i \cong v_i / \sum_{i=1}^m v_i$ . In this case,  $w_i$  can be substituted for  $w_i \cong v_i / \sum_{i=1}^m v_i$ , with

restating the above two properties as (6.7)

$$(a) \sum_{i=1}^m w_i = \sum_{i=1}^m \left( \frac{v_i}{\sum_{i=1}^m v_i} \right) = 1 \quad (b) a_{ij} = \frac{w_i}{w_j} \cong \frac{v_i / \sum_{i=1}^m v_i}{v_j / \sum_{j=1}^m v_j} = \frac{v_i}{v_j} \quad (6.7)$$

According to the change of  $w_i$ , the upper and lower bound constraints of each input and output can also be changed like (6.8).

$$(a) \frac{L_i}{U_i} \leq \frac{v_i}{v_r} = \frac{w_i}{w_r} \leq \frac{U_i}{L_i} \quad (b) \frac{L_r}{U_r} \leq \frac{\mu_r}{\mu_r} = \frac{w_r}{w_r} \leq \frac{U_r}{L_r} \quad (6.8)$$

In the following, we assume that weight results in [29] are obtained based on the relation  $w_i \cong v_i / \sum_{i=1}^m v_i$  satisfying property (6.9). And we showed the results, which are rearranged according to the upper and lower limits.

$$w_r^L = (0.4023, 0.0795, 0.1392, 0.2766), \quad w_r^U = (0.4667, 0.1361, 0.1850, 0.3146)$$

$$w_i^L = (0.75, 0.1667), \quad w_i^U = (0.8333, 0.25)$$

The ratio constraints for applying C/R weight restriction can be made by the following manner. For example, the ratio of weights between  $\mu_1$  and  $\mu_2$  is

$$\frac{0.4023}{0.1361} \leq \frac{\mu_1}{\mu_2} \leq \frac{0.4667}{0.0795} \quad \longrightarrow \quad 2.9559 \leq \frac{\mu_1}{\mu_2} \leq 5.8704$$

All the other ratios among inputs and among outputs can be found as the same way.

Table 6.2 shows the results comparison of two methods.

The CCR model evaluates 7 DMUs as efficient among 12 DMUs, but either method evaluates only DMUs 5 and 7 as efficient, and also we cannot find much difference in the ranking order between two methods. However when W/B weight restriction is applied, all DMU's efficiency scores except DMUs 1 and 2 are appeared slightly lower than those under

C/R weight restriction.

**Table 6.2 Results comparison of C/R and W/B in FMS data**

DMU	CCR	Rank	C/R	Rank	W/B	Rank
1	1	-	0.9734	(5)	0.9750	(4)
2	1	-	0.9504	(6)	0.9714	(5)
3	0.9823	(9)	0.9485	(7)	0.9398	(7)
4	1	-	0.9953	(3)	0.9886	(3)
5	1	-	1	(1)	1	(1)
6	1	-	0.9795	(4)	0.9605	(6)
7	1	-	1	(1)	1	(1)
8	0.9614	(10)	0.9384	(8)	0.8952	(8)
9	1	-	0.8343	(10)	0.1148	(12)
10	0.9535	(11)	0.8303	(11)	0.7863	(11)
11	0.9832	(8)	0.9246	(9)	0.8732	(9)
12	0.8012	(12)	0.7905	(12)	0.7872	(10)

Especially the efficiency score of DMU 9 is 0.1148 under W/B weight restriction, which is far lower than 0.8343 under C/R weight restriction. Like an example 6.1, W/B weight restriction makes very low efficiency score for DMU 9 compared with that under C/R weight restriction.

#### 6.4 Empirical study on the comparison of C/R and W/B weight restrictions

In previous DEA literature, C/R weight restriction was considered to represent 'relative importance' or 'marginal rate of substitution' of variables. Also W/B weight restriction interprets the 'relative importance' as 'proportion of virtual input (output) to total virtual input (output)'. Until now, we compared the characteristics of each weight restriction method using the example of single-input, multiple-output case.

In this section, we performed the empirical study in multiple-input, multiple-output case to compare the characteristics of each method. The purpose of this empirical study is to

compare the results of C/R and W/B weight restrictions in order to see how different results they make when we replace the given W/B weight ratios to C/R weight ratios in the following manner (6.9).

$$\frac{\mu_1 y_1}{\mu_1 y_1 + \mu_2 y_2} = \alpha, \quad \frac{\mu_2 y_2}{\mu_1 y_1 + \mu_2 y_2} = \beta \quad (\text{W/B}) \Rightarrow \frac{\mu_1}{\mu_2} = \frac{\alpha}{\beta} \quad (\text{C/R}) \quad (6.9)$$

For example in one-input, two output case, we compared the results between  $\hat{y}_1(j) = 0.4$ ,  $\hat{y}_2(j) = 0.6$  (W/B) and  $\mu_1 / \mu_2 = 4 / 6$  (C/R).

On the two application examples of FMS and car selection data, each of 6 random weight ratios (case 1 to case 6) is chosen which can make different DMUs to be the 1<sup>st</sup> ranker. It may not acceptable to compare the results just from 6 weight ratios. However from choosing each weight ratio to make different DMUs as 1<sup>st</sup> ranker, we can test wide range of weight vectors. The chosen weight ratios and the detailed results are shown in appendix Table A.6 (FMS) and Table A.7 (Car selection).

Equation (6.10) represents the Spearman coefficient of rank correlations and Table 6.3 reports the nonparametric statistical test of the relationship between rankings under C/R weight restrictions and W/B weight restrictions.

$$r_s = 1 - \frac{6 \sum_{i=1}^n d_i^2}{n(n^2 - 1)} \quad (6.10)$$

where, n = number of observations,  $d_i = S_i - R_i$  ( $S_i$  is the rank under C/R,  $R_i$  is the rank under W/B)

All cases in two applications except *case 6 of FMS* result in the rejection of  $H_0$  at 0.001 level (Spearman). Many of results particularly appeared  $r_s \geq 0.96$  which is very close to 1. This



implies that these two ranking results have a strong direct relationship.

**Table 6.3. Spearman rank statistic for 6 cases**

<b>Applications</b>	<b>Case 1</b>	<b>Case 2</b>	<b>Case 3</b>	<b>Case 4</b>	<b>Case 5</b>	<b>Case 6</b>
FMS	0.972	0.965	0.819	0.958	0.979	0.430
Car Selection	<b>0.978</b>	0.857	0.893	0.964	0.933	0.966

The interesting findings are first, even though these two weight restriction methods are taking different weight vectors to each DMU, the results are not much different in both ranking and the efficiency score of each DMU.

Among the 12 cases, the same DMU is selected as the 1<sup>st</sup> ranker in 11 cases. Also in 3 cases under W/B weight restriction, two or three DMUs get the efficiency score 1.00 and ranked 1<sup>st</sup> which are also high rankers under C/R weight restriction.

Second, the considerable difference is found, which may be one of the extreme cases, in case 6 of FMS data. While DMU 9 is selected as 1<sup>st</sup> ranker under C/R but last under W/B weight restriction. It is due to the reason that C/R allows the rather flexible substitution among variables while W/B doesn't allow the flexible substitution among variables.

For DMU 9 ( $j = 9$ ), when we calculate the W/B proportion of each variable using CCR multipliers, the result is  $\hat{x}_1(j) = 0.38$ ,  $\hat{x}_2(j) = 0.62$ ,  $\hat{y}_1(j) = 0$ ,  $\hat{y}_2(j) = 0$ ,  $\hat{y}_3(j) = 0$ ,  $\hat{y}_4(j) = 1$ . Therefore we can see that DMU 9 can get the highest W/B efficiency score when only producing  $\hat{y}_4(j)$ . However, not only the zero proportion under W/B weight restriction cannot be defined but also unrealistic, we tested the following three cases of W/B weight restriction for DMU 9 in which input proportions are kept the same as before.

$$\text{Case (1) : } \hat{y}_1(j) = 0.005, \hat{y}_2(j) = 0.005, \hat{y}_3(j) = 0.005, \hat{y}_4(j) = 0.985$$

$$\text{Case (2) : } \hat{y}_1(j) = 0.010, \hat{y}_2(j) = 0.010, \hat{y}_3(j) = 0.010, \hat{y}_4(j) = 0.970$$

Case (3) :  $\hat{y}_1(j) = 0.030$ ,  $\hat{y}_2(j) = 0.030$ ,  $\hat{y}_3(j) = 0.030$ ,  $\hat{y}_4(j) = 0.910$

The W/B efficiency score of DMU 9 is dramatically decreased by increasing output proportions of  $\hat{y}_1(j)$ ,  $\hat{y}_2(j)$ ,  $\hat{y}_3(j)$ . The result of W/B efficiency score (DMU 9) is 0.707 (Case 1), 0.536 (Case 2), 0.262 (Case 3) which shows that DMU 9 is the one of the extreme cases in view of W/B weight restriction. We also tested the above three cases with changing input proportions, but the efficiency score cannot exceed 0.3. This represents the above fact that C/R allows rather flexible substitution among variables while W/B almost doesn't allow substitution among variables.

## 6.5 Conclusions

In this chapter, we analyzed the characteristics of W/B weight restrictions theoretically and compared with those of C/R weight restriction empirically. We showed that under W/B weight restriction, each DMU takes all different weight vectors and some DMUs may have limiting efficiency score.

To see the practical difference between two restriction methods, we performed empirical study and showed that even though C/R and W/B weight restriction take different weight vectors for each DMU, ranking results of empirical study are appeared to be very similar in many cases of multiple-input, multiple-output situation. However, the empirical study also demonstrated that the ranking result can be far different when a certain DMU has limiting efficiency score under W/B weight restriction. This is based on the fact that while C/R weight restriction allows flexible substitution among inputs or outputs, W/B weight restriction would not allow this flexible substitution.

## **CHAPTER 7. ALTERNATIVE APPROACH TO MEASURE OVERALL EFFICIENCY**

### **7.1 Introduction**

In this chapter, we present alternative models, which can measure each of overall efficiency (OE) with cone-ratio weight restrictions and compare with previous models using examples.

The importance of measuring overall efficiency has been emphasized in DEA literature [11], [36], [38]. Sueyoshi [36] indicated that when the goal of each DMU is to achieve the lowest input prices, measuring allocative efficiency (AE) is much more important than achieving technical efficiency (TE).

However the research for measuring overall (allocative) efficiency (OE, AE) has been rather limited. This is mainly due to the belief that it can be measured only when the information on prices and costs are exactly known, but to get this exact information in real applications is not easy. Cooper et al [11] mentioned the problems for measuring overall (allocative) efficiency in real applications such that many companies are unwilling to disclose their unit costs and unit prices may also be a problem when these values are subject to large fluctuations. Also when the decision maker's interest is not limited only to cost or price, there are many factors that cannot be easily quantifiable, for example variables represented by quality or customer service parameters.

The previous models for measuring overall (allocative) efficiency uses a two-step approach. For instance, to measure the cost efficiency, it tries to find the optimal quantities of each input of DMU  $j$  with objective function which minimizes the actual total costs in step 1

and in step 2 it calculates the overall efficiency by the ratio of total optimal costs to total actual costs of DMU  $j$ . Therefore it is believed that when we don't have exact information on prices or costs, we cannot perform the calculation in step 1, so we cannot perform the calculation in step 2 either. This belief made even harder to use previous two-step models effectively in the case that we need to do some efficiency analysis according to possible ranges of prices or costs.

In this chapter, we developed the models that can measure the overall efficiency in a single step. The only difference between the proposed and CCR model is the added cost (price) vector constraints, which results in the DEA models with cone-ratio weight restrictions. That is, the proposed model can directly measure the overall efficiency score of DMU  $j$ .

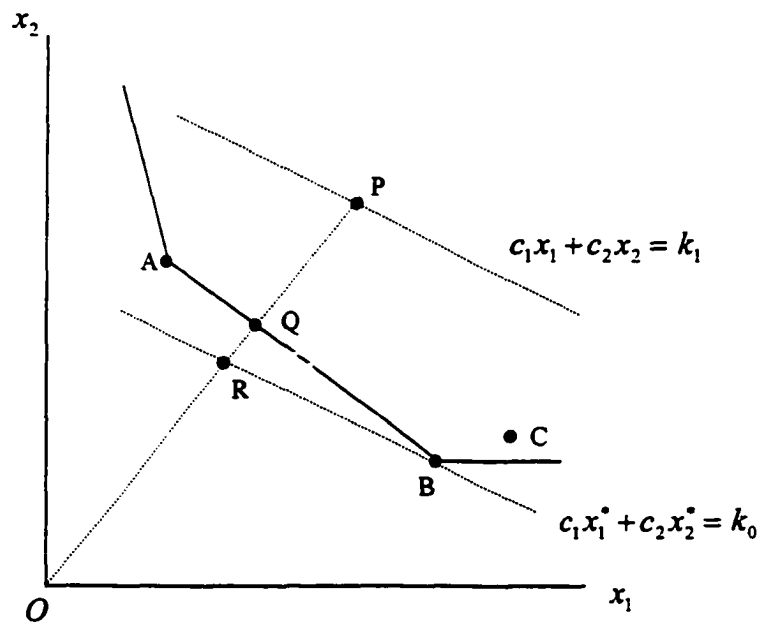
In previous DEA literature, we couldn't find any prior reference in which the relationship between two models (models for measuring overall (allocative) efficiency and the models with cone-ratio weight restrictions) is explored and thus these two models have been generally treated separately. However, through the suggested models, we can show the relationship between two models such that the models for measuring overall efficiency can be considered as a subset of the models with general cone-ratio restrictions.

The rest of this chapter is organized as follows. In section 7.2, we introduce the concepts of three efficiency measures in DEA. In section 7.3, previous models for measuring overall efficiency are presented with examples. And in section 7.4, alternative models for measuring overall efficiency are developed and compared with previous models using examples. Finally, conclusions are provided in section 7.5.

We want to indicate that many descriptions and models in section 7.2 and 7.3 are referenced by recently published DEA text by Cooper et al [11].

## 7.2 The concepts of three efficiency measures

Figure 7.1 shows the concepts of three efficiency measures originated by Farrell. Let us assume that 1) each DMU uses two inputs ( $x_1, x_2$ ) in order to yield a single output ( $y$ ), under the condition of constant returns to scale 2) two inputs and one output are assumed to be all positive.



**Figure 7.1 Technical, Allocative and Overall efficiency**

P is a point in the interior of the production possibility set representing the activity of a DMU which produces this same amount of output but with greater amounts of both inputs than any point on the production frontier.

Then three efficiency measures can be defined as follows. First, technical efficiency ( $TE$ ) of DMU  $P$  can be measured by  $TE = OQ / OP$  since DMU  $Q$  exists on the frontier that

use less input quantities than  $P$  to yield the same quantity of output. Second, allocative efficiency ( $AE$ ) of DMU  $P$  can be measured by  $AE = OR / OQ$ . The budget (cost) line, which has the slope equal to the ratio of two input prices of DMU  $P$  is  $c_1x_1 + c_2x_2 = k_1$  and that of DMU  $B$  is  $c_1x_1^* + c_2x_2^* = k_0$  ( $k_0 < k_1$ ). Therefore the cost of  $(k_1 - k_0)$  can be reduced by moving this line in parallel fashion until it intersects with the isoquant at  $B$ . The corresponding measure of  $(1 - OR / OQ)$  indicates the allocative inefficiency that denotes a possible reduction in cost by using appropriate input mixes. Third, overall efficiency ( $OE$ ) denotes a possible reduction in cost due to changing from  $P$  (observed input quantities) to  $B$  (cost minimizing input quantities) and it can be measured by  $OE = OR / OP$ .

Therefore we have the relation to each of three efficiency measures

$$OE = \frac{OR}{OP} = \frac{OQ}{OP} \times \frac{OR}{OQ} = TE \times AE \quad (7.1)$$

### 7.3 Previous models for measuring overall efficiency

#### 7.3.1 Performing procedure

The concept of overall efficiency has been researched focusing on each preferable interest by several ways, i.e. 1) cost efficiency, 2) revenue efficiency, 3) profit efficiency and 4) ratio efficiency. This standard approach to determine overall efficiency and its components is due to Färe et al [16].

In this section, we begin with explanation of the following models that have been used to measure overall efficiencies in DEA literature. These models are well introduced in recently published DEA text [11]. To calculate each overall efficiency, we have to perform

the following two-step procedures.

Step 1. From the following LP models, we can find the optimal  $x_i^*$  and  $y_r^*$  for each

DMU  $j$ . Where,  $c_{io}$  is the unit cost of the input  $x_i$  of DMU<sub>0</sub>, and  $p_{ro}$  is the unit price of the output  $y_r$  of DMU<sub>0</sub>, which may vary from one DMU to another.

$$\begin{aligned}
 \text{(Cost-E)} \quad & \text{Min} \quad \sum_{i=1}^m c_{io} x_i \\
 \text{subject to} \quad & x_i \geq \sum_{j=1}^n x_{ij} \lambda_j, \quad i = 1, \dots, m \\
 & y_{ro} \leq \sum_{j=1}^n y_{rj} \lambda_j, \quad r = 1, \dots, s \\
 & \lambda_j \geq 0, \quad \forall j
 \end{aligned} \tag{7.2-a}$$

$$\begin{aligned}
 \text{(Revenue-E)} \quad & \text{Max} \quad \sum_{r=1}^s p_{ro} y_r \\
 \text{subject to} \quad & x_{io} \geq \sum_{j=1}^n x_{ij} \lambda_j, \quad i = 1, \dots, m \\
 & y_r \leq \sum_{j=1}^n y_{rj} \lambda_j, \quad r = 1, \dots, s \\
 & \lambda_j \geq 0, \quad \forall j
 \end{aligned} \tag{7.3-a}$$

$$\begin{aligned}
 \text{(Profit-E)} \quad & \text{Max} \quad \sum_{r=1}^s p_{ro} y_r - \sum_{i=1}^m c_{io} x_i \\
 \text{subject to} \quad & x_i = \sum_{j=1}^n x_{ij} \lambda_j \leq x_{io}, \quad i = 1, \dots, m \\
 & y_r = \sum_{j=1}^n y_{rj} \lambda_j \geq y_{ro}, \quad r = 1, \dots, s \\
 & \lambda_j \geq 0, \quad \forall j
 \end{aligned} \tag{7.4-a}$$

$$\begin{aligned}
 \text{(Ratio-E)} \quad & \text{Max} \quad \frac{\sum_{r=1}^s p_{ro} y_r}{\sum_{i=1}^m c_{io} x_i} \\
 \text{subject to} \quad & x_i = \sum_{j=1}^n x_{ij} \lambda_j \leq x_{io}, \quad i = 1, \dots, m
 \end{aligned}$$

$$y_r = \sum_{j=1}^n y_{rj} \lambda_j \geq y_{ro}, \quad r = 1, \dots, s \quad (7.5-a)$$

$$\lambda_j \geq 0, \quad \forall j$$

Step 2. Using the optimal value of  $x_i^*$  and  $y_r^*$ , in step 2, we can calculate each overall efficiency for DMU  $j$ .

$$E_C = \frac{\sum_{i=1}^m c_{io} x_i^*}{\sum_{i=1}^m c_{io} x_{io}}, \quad (7.2-b)$$

$$E_R = \frac{\sum_{r=1}^s p_{ro} y_{ro}}{\sum_{r=1}^s p_{ro} y_r^*}, \quad (7.3-b)$$

$$E_P = \frac{\sum_{r=1}^s p_{ro} y_{ro} - \sum_{i=1}^m c_{io} x_{io}}{\sum_{r=1}^s p_{ro} y_r^* - \sum_{i=1}^m c_{io} x_i^*}, \quad (7.4-b)$$

$$E_{Ratio} = \frac{\sum_{r=1}^s p_{ro} y_{ro} / \sum_{i=1}^m c_{io} x_{io}}{\sum_{r=1}^s p_{ro} y_r^* / \sum_{i=1}^m c_{io} x_i^*}, \quad (7.5-b)$$

### 7.3.2 Example

Here we suggest an example 7.1, which is excerpted from Cooper et al [11] (p.247) for the purpose of explaining the above models and afterward comparing the result with those of our models. Table 7.1 shows the data for 4 DMUs with two inputs and two outputs, along with the unit cost for each input and unit price for each output. And Table 7.2 shows the results of Cost, Revenue, Profit and Ratio efficiencies of the 4 DMUs.

**Table 7.1 Example 7.1 data**

DMU	Input		Output		Input Cost		Output Price	
	$x_1$	$x_2$	$y_1$	$y_2$	$c_1$	$c_2$	$p_1$	$p_2$
1	2	3	5	8	2	2	5	5
2	1	5	2	6	2	4	4	5
3	3	8	4	8	3	3	6	4
4	2	7	1	2	4	2	7	4



According to the above models, for example, cost efficiency for DMU 2 is calculated by the following procedure.

Step 1. Solve the model (2-a)

$$\begin{aligned}
 \text{(Cost-E)} \quad & \text{Min} \quad 2x_1 + 4x_2 \\
 \text{subject to} \quad & x_1 - 2\lambda_1 - 1\lambda_2 - 3\lambda_3 - 2\lambda_4 \geq 0 \\
 & x_2 - 3\lambda_1 - 5\lambda_2 - 8\lambda_3 - 7\lambda_4 \geq 0 \\
 & 5\lambda_1 + 2\lambda_2 + 4\lambda_3 + 1\lambda_4 \geq 2 \\
 & 8\lambda_1 + 6\lambda_2 + 8\lambda_3 + 2\lambda_4 \geq 6 \\
 & \lambda_j \geq 0, \quad \forall j
 \end{aligned}$$

Then we get the optimal solution such that  $x_1 = 1.5$ ,  $x_2 = 2.25$ ,  $\lambda_1 = 0.75$  with objective function value 12. All the other  $\lambda$  variables are 0. (i.e.  $\lambda_2 = \lambda_3 = \lambda_4 = 0$ )

Step 2. Solve the model (2-b)

$$E_C = \frac{\sum_{i=1}^m c_{io} x_i^*}{\sum_{i=1}^m c_{io} x_{io}} = \frac{(2 \times 1.5) + (4 \times 2.25)}{(2 \times 1) + (4 \times 5)} = \frac{12}{22} = 0.545$$

**Table 7.2 Results of example 7.1**

DMU	CCR	Cost	Revenue	Profit	Ratio
1	1	1	1	1	1
2	1	0.545	1	1	0.461
3	0.571	0.455	0.571	0.326	0.411
4	0.214	0.159	0.208	0	0.142

The revenue, profit and ratio efficiencies are also can be obtained by solving sequentially the model (3-a, 3-b), (4-a, 4-b) and (5-a, 5-b) respectively. When the actual profit appears to be negative, the profit efficiency score is assigned the value 0.

## 7.4 Alternative approach to measure overall efficiency

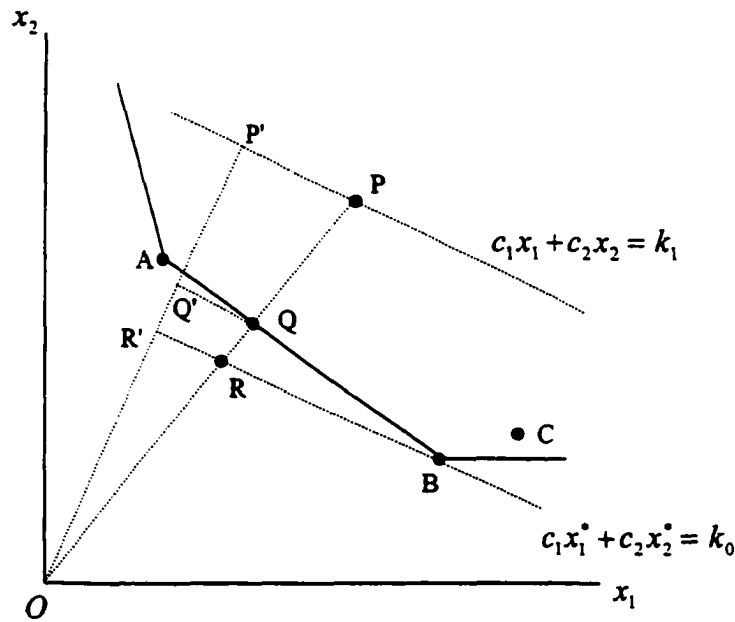
### 7.4.1 The alternative models using cone-ratio constraints

In this section, we suggest the models, which can measure the overall efficiency (i.e. cost efficiency, revenue efficiency and ratio efficiency) using cone-ratio constraints. Using Figure 7.1, we showed the concepts of three efficiency measures of DMU  $P$  are

$$TE = OQ / OP, \quad AE = OR / OQ, \quad OE = OR / OP.$$

And the relationship of three efficiency measures is

$$OE = \frac{OR}{OP} = \frac{OQ}{OP} \times \frac{OR}{OQ} = TE \times AE.$$



**Figure 7.2 Illustration of suggested model**

Also  $c_1x_1 + c_2x_2 = k_1$  and  $c_1x_1^* + c_2x_2^* = k_0$  are isocost lines for DMU B and DMU P respectively which are parallel to each other. In Figure 7.2,  $OP'$  represents an orthogonal vector to the isocost lines, which passes through the origin. It is clear from above figure that we can find the unique vector, which is perpendicular to the isocost lines and passes through

the origin. And  $Q'$ ,  $R'$  are the projection points which are projected perpendicular to the vector  $OP'$  from  $Q$  and  $R$  respectively. Since  $Q$  and  $Q'$ ,  $R$  and  $R'$ ,  $P$  and  $P'$  lie on each isocost line, it is clear that these points have the same costs respectively.

Therefore, the following relation (7.6) should hold that is also obvious by the property of right-angled triangle in  $\Delta OPP'$ .

$$TE = \frac{OQ}{OP} = \frac{OQ'}{OP'}, \quad AE = \frac{OR}{OQ} = \frac{OR'}{OQ'}, \quad OE = \frac{OR}{OP} = \frac{OR'}{OP'} \quad (7.6)$$

And the relationship of three efficiency measures also holds by (7.7)

$$OE = \frac{OR'}{OP'} = \frac{OQ'}{OP'} \times \frac{OR'}{OQ'} = TE \times AE \quad (7.7)$$

And the properties suggested in chapter 5 can be modified as follows.

We define the vector  $OP'$  as the cost vector as follows (D1), (D2).

(D1) Cost vector  $\bar{c}$  is a vector, which is perpendicular to the iso-cost lines (planes) of

DMU  $j$  and passes through the origin. Similarly, price vector  $\bar{p}$  is a vector, which is perpendicular to the iso-revenue lines (planes) of DMU  $j$  and passes through the origin.

(D2) The cost vector of DMU  $j$ , which uses  $m$  inputs with the unit costs  $(c_1, \dots, c_m)$  and

passes through the origin, is the  $(c_1, \dots, c_m)$ . Similarly, the price vector of DMU  $j$ , which produces  $r$  outputs with the unit prices  $(p_1, \dots, p_r)$  and passes through the origin, is the  $(p_1, \dots, p_r)$ .

After all, when the cone-ratio weight restrictions (here, cost or price ratios) are applied to the general CCR model, all DMUs are projected to the cost (price) vector along with the iso-cost (iso-revenue) lines (planes), and the overall efficiency is measured by the following

ratio (P1).

(P1) When the cost (price) vectors are applied to the CCR model, the overall efficiency of

DMU  $j$  can be measured by the following ratio.

(one-input, multiple output or multiple-input, one output case)

*Overall efficiency score of DMU  $j$  =*

$$\frac{(\text{Norms of orthogonal projection of DMU } j \text{ to the cost (price) vector})}{(\text{Norms of orthogonal projection of DMU } j^* \text{ to the cost (price) vector})}$$

where, DMU  $j^*$  has the largest norm (revenue maximization case) or smallest

norm (cost minimization case) when projected to the price (cost) vector.

However in multiple-input, multiple-output cases, we cannot represent the efficiency of DMU  $j$  using 2-dimensional figure and thus we have to say that

*Overall efficiency score of DMU  $j$  =*

$$\frac{\text{virtual revenue of DMU } j / \text{virtual costs of DMU } j}{\text{virtual revenue of DMU } j^* / \text{virtual costs of DMU } j^*}$$

where, DMU  $j^*$  has the largest efficiency score with the same input and output

cost (price) vectors with DMU  $j$  respectively.

The weight vector for measuring overall efficiency can be represented such as cost vector :  $\bar{c} = (v_1, \dots, v_m) = (c_1, \dots, c_m)$ , price vector :  $\bar{p} = (\mu_1, \dots, \mu_r) = (p_1, \dots, p_r)$ . Therefore when we consider the DEA models for measuring overall efficiency as one of general cone-ratio restrictions, we can replace the each of input, output multiplier to each corresponding cost and price. Therefore, the following two properties (7.8), (7.9) hold.

$$\frac{v_1}{v_2} = \frac{c_1}{c_2}, \dots, \frac{v_1}{v_m} = \frac{c_1}{c_m}, \frac{v_2}{v_3} = \frac{c_2}{c_3}, \dots, \frac{v_{m-1}}{v_m} = \frac{c_{m-1}}{c_m} \quad (7.8)$$

$$\frac{\mu_1}{\mu_2} = \frac{p_1}{p_2}, \dots, \frac{\mu_1}{\mu_s} = \frac{p_1}{p_s}, \frac{\mu_2}{\mu_3} = \frac{p_2}{p_3}, \dots, \frac{\mu_{s-1}}{\mu_s} = \frac{p_{s-1}}{p_s} \quad (7.9)$$

That is, the only difference of proposed model with CCR model is the added constraints according to the each case as follows.

1. Cost Efficiency (7.10) : add constraint (7.8)

$$\begin{aligned} \text{( Cost-E )} \quad \text{Max}_{u_r} \quad h_{j_0} &= \sum_{r=1}^s \mu_r y_{rj_0} \\ \text{subject to} \quad \sum_{i=1}^m v_i x_{ij_0} &= 1 \\ \sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} &\leq 0, \quad j = 1, \dots, n \\ \frac{v_1}{v_2} &= \frac{c_1}{c_2}, \dots, \frac{v_1}{v_m} = \frac{c_1}{c_m}, \\ \mu_r, v_i &\geq 0, \quad \forall r \text{ and } i \end{aligned} \quad (7.10)$$

2. Revenue Efficiency (7.11) : add constraint (7.9)

$$\begin{aligned} \text{( Revenue-E )} \quad \text{Max}_{u_r} \quad h_{j_0} &= \sum_{r=1}^s \mu_r y_{rj_0} \\ \text{subject to} \quad \sum_{i=1}^m v_i x_{ij_0} &= 1 \\ \sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} &\leq 0, \quad j = 1, \dots, n \\ \frac{\mu_1}{\mu_2} &= \frac{p_1}{p_2}, \dots, \frac{\mu_1}{\mu_s} = \frac{p_1}{p_s}, \\ \mu_r, v_i &\geq 0, \quad \forall r \text{ and } i \end{aligned} \quad (7.11)$$

3. Ratio Efficiency (7.12) : add constraints (7.8) and (7.9).

$$\begin{aligned}
\text{( Ratio-E )} \quad & \text{Max}_{\mu_r} h_{j_o} = \sum_{r=1}^s \mu_r y_{rj_o} \\
\text{subject to} \quad & \sum_{i=1}^m v_i x_{ij_o} = 1 \\
& \sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, n \\
& \frac{v_1}{v_2} = \frac{c_1}{c_2}, \dots, \frac{v_1}{v_m} = \frac{c_1}{c_m}, \\
& \frac{\mu_1}{\mu_2} = \frac{p_1}{p_2}, \dots, \frac{\mu_1}{\mu_s} = \frac{p_1}{p_s}, \\
& \mu_r, v_i \geq 0, \quad \forall r \text{ and } i
\end{aligned} \tag{7.12}$$

From the property of (7.8), the following formulation (7.13) is equivalent to (7.10).

$$\begin{aligned}
\text{( Cost-E )} \quad & \text{Max}_{p_r} h_{j_o} = \sum_{r=1}^s p_r y_{rj_o} \\
\text{subject to} \quad & \sum_{i=1}^m c_i x_{ij_o} = 1 \\
& \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} \leq 0, \quad j = 1, \dots, n \\
& \frac{c_1}{c_2} = \frac{v_1}{v_2}, \dots, \frac{c_1}{c_m} = \frac{v_1}{v_m}, \\
& p_r, c_i \geq 0, \quad \forall r \text{ and } i
\end{aligned} \tag{7.13}$$

Similarly we can make formulations which is equivalent to (7.11) and (7.12) respectively with replacing  $v_i$  to  $c_i$ ,  $\forall i$  and  $\mu_r$  to  $p_r$ ,  $\forall r$ .

After all, the previous models are equivalent to the suggested models that can be proved as follows. For example, when we think of cost efficiency of DMU  $j_o$ , the previous model uses the two-step approach (7.2-a) and (7.2-b).

Now we think of the dual of (7.2-a), it becomes (7.14).

$$\begin{aligned}
& \text{Max} \quad \sum_{r=1}^s \mu_r y_{ro} \\
\text{subject to} \quad & \sum_{i=1}^m v_i x_i = 1 \\
& \sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad \forall j \\
& 0 \leq v_i \leq c_{io}, \quad i = 1, \dots, m \\
& \mu_r, v_i \geq 0, \quad \forall r \text{ and } i
\end{aligned} \tag{7.14}$$

By primal-dual relationships, two properties should be hold

$$1) \sum_{r=1}^s \mu_r^* y_{ro} = \sum_{i=1}^m c_{io} x_i^*$$

$$2) \text{ from complementary slackness condition, } x_i^* (c_{io} - v_i) = 0.$$

That is, if  $x_i^* > 0$ , then  $v_i = c_{io}$ ,  $\forall i$  at optimality

Therefore, (7.14) can be rewritten as (7.15) at optimality

$$\begin{aligned}
& \text{Max} \quad \sum_{r=1}^s \mu_r y_{ro} \\
\text{subject to} \quad & \sum_{i=1}^m c_{io} x_i = 1 \\
& \sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m c_{io} x_{ij} \leq 0, \quad \forall j \\
& \mu_r \geq 0, \quad \forall r
\end{aligned} \tag{7.15}$$

After all, cost efficiency of DMU  $j_o$  in previous model (7.2-b) can be calculated by

suggested alternative model (7.10) by reason of (7.16)

$$E_C = \frac{\sum_{i=1}^m c_{io} x_i^*}{\sum_{i=1}^m c_{io} x_{io}} = \frac{\sum_{r=1}^s \mu_r^* y_{ro}}{\left( \sum_{r=1}^s \mu_r y_{rj} \right)_{\max}} \tag{7.16}$$

Also, equivalence of models for revenue and ratio efficiency of DMU  $j$  can be proved

in a similar manner.

#### 7.4.2 Example

Here, we explain above models using two examples 1) when the unit costs (prices) are the same for all DMUs (Example 7.2) 2) when the unit costs (prices) are not the same for all DMUs (Example 7.1). Table 7.3 shows the data of first case and the data of second case was shown in Table 7.2. It is clear in Figure 7.3 that DMU 1 and 2 are CCR-efficient but only DMU 2 is overall efficient (here, cost efficient).

**Table 7.3. Example 7.2 data and results**

DMU	$x_1$	$x_2$	$c_1$	$c_2$	DMU	CCR	Cost-E
1	4	2	4	2	1	1	0.8
2	2	4	4	2	2	1	1
3	4	6	4	2	3	0.6	0.571

Cost efficiency of DMU 3 is explained as a ratio  $OR / O3$ , but suggested model measures the ratio  $OR' / O3'$ . Both measures should be the same from the property of right-angled triangle.

Model to find the cost efficiency of DMU 3 is

$$\begin{aligned}
 & \text{(Cost-E)} \quad \text{Max } \mu \\
 & \text{subject to} \quad 4v_1 + 6v_2 = 1 \\
 & \quad \quad \quad \mu - 4v_1 - 2v_2 \leq 0 \\
 & \quad \quad \quad \mu - 2v_1 - 4v_2 \leq 0 \\
 & \quad \quad \quad \mu - 4v_1 - 6v_2 \leq 0 \\
 & \quad \quad \quad v_1 - 2v_2 = 0 \\
 & \quad \quad \quad \mu \geq 0, \quad v_1 \geq 0, \quad v_2 \geq 0
 \end{aligned}$$



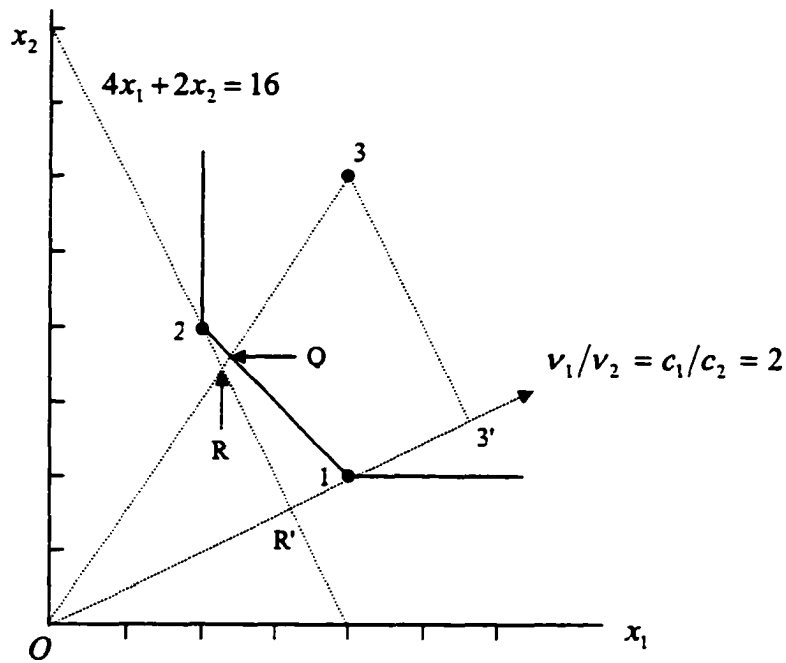


Figure 7.3. Illustration of example 7.2

And it gives the solution  $h_{j_0}^* = \theta^* = 0.57143$ . When we multiply the efficiency score to the amount of each input of DMU 3,  $\theta^*(x_1, x_2) = 0.57143 \times (4, 6) = (2.2857, 3.42858)$ , then this point is the coordinate of point  $R$ . Therefore we can see that the above model measures the ratio of  $OR / O3 = OR' / O3'$ . On the other hand, when we use the model (7.2-a) and (7.2-b) to measure cost efficiency of DMU 3, we also can obtain the same score. That is,

$$\begin{aligned}
 \text{(Cost-E)} \quad & \text{Min } 4x_1 + 2x_2 \\
 \text{subject to} \quad & x_1 - 4\lambda_1 - 2\lambda_2 - 4\lambda_3 \geq 0 \\
 & x_2 - 2\lambda_1 - 4\lambda_2 - 6\lambda_3 \geq 0 \\
 & \lambda_1 + \lambda_2 + \lambda_3 \geq 1 \\
 & \lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_3 \geq 0.
 \end{aligned}$$

And it gives the solution with  $x_1^* = 2$ ,  $x_2^* = 4$ ,  $\lambda_2 = 1$  and  $\lambda_1 = \lambda_3 = 0$ , which is the same with the coordinate of DMU 2. Therefore

$$E_c = \frac{\sum_{i=1}^m c_{io} x_i^*}{\sum_{i=1}^m c_{io} x_{io}} = \frac{(4 \times 2) + (2 \times 4)}{(4 \times 4) + (2 \times 6)} = \frac{16}{28} = 0.57143$$

After all in this example 7.2, we can state the difference in the way of measuring overall efficiency between two models as follows. That is, to measure the cost efficiency of DMU 3, proposed model measures the ratio of  $OR / O3 = OR' / O3'$  directly while previous model measures the ratio of total costs of DMU 2 / total costs of DMU 3.

Then the allocative efficiency ( $AE$ ) of DMU 3 can be obtained from equation (7.7)

$$AE = \frac{OE}{TE} = \frac{E_c}{TE} = \frac{0.571}{0.6} = 0.952$$

When the unit costs (prices) are not the same for all DMUs like example 7.1, we can also apply models (7.10), (7.11) and (7.12) replacing previous models (7.2), (7.3) and (7.5) respectively.

1) Cost efficiency of DMU 2 : objective function value = 0.54545

$$\text{(Cost-E)} \quad \text{Max} \quad 2\mu_1 + 6\mu_2$$

$$\text{subject to} \quad 1v_1 + 5v_2 = 1$$

$$5\mu_1 + 8\mu_2 - 2v_1 - 3v_2 \leq 0$$

$$2\mu_1 + 6\mu_2 - 1v_1 - 5v_2 \leq 0$$

$$4\mu_1 + 8\mu_2 - 3v_1 - 8v_2 \leq 0$$

$$1\mu_1 + 2\mu_2 - 2v_1 - 7v_2 \leq 0$$

$$2v_1 - v_2 = 0 \quad \longrightarrow \quad \text{cost vector constraint}$$

$$\mu_1 \geq 0, \quad \mu_2 \geq 0, \quad v_1 \geq 0, \quad v_2 \geq 0$$

2) Revenue efficiency of DMU 3 : objective function value = 0.57143

$$\text{(Revenue-E)} \quad \text{Max} \quad 4\mu_1 + 8\mu_2$$

$$\begin{aligned}
\text{subject to} \quad & 3v_1 + 8v_2 = 1 \\
& 5\mu_1 + 8\mu_2 - 2v_1 - 3v_2 \leq 0 \\
& 2\mu_1 + 6\mu_2 - 1v_1 - 5v_2 \leq 0 \\
& 4\mu_1 + 8\mu_2 - 3v_1 - 8v_2 \leq 0 \\
& 1\mu_1 + 2\mu_2 - 2v_1 - 7v_2 \leq 0 \\
& 4\mu_1 - 6\mu_2 = 0 \longrightarrow \text{price vector constraint} \\
& \mu_1 \geq 0, \mu_2 \geq 0, v_1 \geq 0, v_2 \geq 0
\end{aligned}$$

3) Ratio efficiency of DMU 3 : objective function value = 0.41056

$$\begin{aligned}
\text{(Ratio-E)} \quad & \text{Max} \quad 4\mu_1 + 8\mu_2 \\
\text{subject to} \quad & 3v_1 + 8v_2 = 1 \\
& 5\mu_1 + 8\mu_2 - 2v_1 - 3v_2 \leq 0 \\
& 2\mu_1 + 6\mu_2 - 1v_1 - 5v_2 \leq 0 \\
& 4\mu_1 + 8\mu_2 - 3v_1 - 8v_2 \leq 0 \\
& 1\mu_1 + 2\mu_2 - 2v_1 - 7v_2 \leq 0 \\
& 3v_1 - 3v_2 = 0 \longrightarrow \text{cost vector constraint} \\
& 4\mu_1 - 6\mu_2 = 0 \longrightarrow \text{price vector constraint} \\
& \mu_1 \geq 0, \mu_2 \geq 0, v_1 \geq 0, v_2 \geq 0
\end{aligned}$$

Even though we showed just one case in each of efficiency, all solutions are exactly equal to the result in Table 7.2. When the unit costs (prices) are not the same for all DMUs, we should use the DMU's own cost vector (price vector) which is being analyzed.

## 7-5. Conclusions

In this chapter, we developed alternative models for measuring overall efficiency (OE) with cone-ratio weight restrictions and compared those with previous models.

The contributions by suggesting alternative models we believe are

1) We showed that the overall efficiency (cost / revenue / ratio efficiency) model can be

referred as one of the general cone-ratio restriction type problems, which considers only prices of outputs or costs of inputs as applied weights.

2) From suggested models, it is clear that all we need to know to measure the overall efficiency is not the exact prices or costs but the ratios of prices or costs. Therefore in cases where we have insufficient information on prices or costs, we can extend the availability of these models by applying possible ranges of price or cost ratios.

Although models for measuring overall efficiency can give some useful information, there still needs a further research to overcome the following. First, these models do not identify a specific way for increasing overall efficiency, but they indicate optimal ratios of costs (prices). Second, the overall efficiency of DMU  $j$  is calculated based on the assumption that all the other DMUs are assumed to use DMU  $j$ 's cost or price vectors, and it may not appropriately reflect a variety of management strategies.

## **CHAPTER 8. CONCLUSIONS**

### **8.1 Summary**

Other than measuring relative efficiency, DEA has been used in a number of other ways to elaborate further on the performance of individual units or to ascertain how the units could become more efficient. Also researchers have developed methods for using DEA as a ranking model, which results in a recent review by Adler et al [1].

In this dissertation, we classified DEA ranking models into two categories based on whether preferences (weights) are given or not. In fact, many DEA ranking models start with the assumption that there are no given preferences (criteria), which is often the case in real life applications.

When the decision maker's preferences (weights) are not given, the ranking criteria and corresponding results of each model vary by the methods each model uses. When decision maker's preferences (weights) are given, the accuracy and acceptability of the results depend on how well these given preferences are reflected to each weight restriction method. This motivates the research on the characteristics of each model, which hopefully can help decision makers to make a better decision.

In chapter 3 and 4, we analyzed the characteristics of A-P model and cross-efficiency evaluation, which are frequently used in case that we don't have any prior relative weights of inputs and outputs. In chapter 5 and 6, we considered the characteristics of DEA ranking models with cone-ratio and Wong and Beasley weight restrictions, both of which take decision maker's preferences into account each of weight restrictions. Finally in chapter 7, we suggested alternative models for measuring overall efficiency.

The followings are a summary of conclusions of each chapter.

In chapter 1, we suggested classification of DEA ranking models and purposes of using DEA ranking models. This is done to say that the purpose of DEA ranking models (even though they have the name of ranking model) can go beyond ranking to importing additional useful information. This information could be helpful in selecting DEA ranking models according to one's purposes.

In chapter 2, we briefly introduced the CCR model with basic definitions, units invariance theorem and example as well as the weight restriction models suggested in previous DEA literature.

In chapter 3, to identify the characteristics of the A-P model and cross-efficiency evaluation, we provided empirical ranking results in both models after describing their ranking criteria. Then we suggested a specialization index (SI) in the A-P model and a  $A_k$  score in the cross-efficiency evaluation to identify specialized DMU. The result table used to find the SI score clearly shows A-P model characteristics. That is, the A-P model often selects the 1<sup>st</sup> ranker among specialized performers, which doesn't have any near 2<sup>nd</sup> follower.

Also we examined the primary conclusions on the 1<sup>st</sup> ranker of cross-efficiency evaluation and showed these conclusions are not always true. That is, the 1<sup>st</sup> ranker in cross-efficiency evaluation is not always the winner with many competitors in the sample.

Finally we suggested the biplot, which facilitates the comparison of characteristics of each model visually. Based on the fact that cross-efficiency evaluation uses almost fixed weights in many of multiple-input, multiple-output cases, we can represent the weight direction of cross-efficiency evaluation in the biplot.

Empirical studies are performed to compare the 1<sup>st</sup> ranker in cross-evaluation with that under restriction of equal input, output weights along with the explanation of simple case in which cross-efficiency makes unexpected ranking result.

In chapter 4, we showed that cross-efficiency evaluation in effect applies almost fixed weights in many of multiple-input, multiple-output cases, which is done as an extension of previous work [4] that focused on single-input, multiple-outputs case.

The contributions in this chapter are first, we showed that when the input values of all DMUs are not unified 1, cross-efficiency scores are not exactly the same as those under fixed weighting scheme. Second, we developed an equation, which shows an efficiency score under fixed weighting scheme in multiple-input, multiple-output situation. Third, we analyzed the difference between real cross-efficiency scores and those under fixed weighting scheme in multiple-input, multiple-output situation by analytically and empirically. Empirical results showed that cross-efficiency evaluation in effect applies almost fixed weights in many of multiple-input, multiple-output cases.

In chapter 5, we proved two properties (P1) and (P2). Property (P1) shows a way to measure the efficiency score when cone-ratio weight restrictions are applied under constant returns to scale with single- input, multiple-outputs (or multiple-inputs, single-output) in DEA. And property (P2) indicates that in multiple-input, multiple output case, C/R weight ratios don't represent perfect substitution among inputs (or outputs) unlike in single-input, multiple-output case.

Based on property (P1), we proposed some graphical explanations of other DEA issues, 1) multiple solution problem 2) multipliers of cross-efficiency evaluation 3) target points under cone-ratio weight restrictions using one-input, two-output case in DEA. We believe

that a graphical explanation can be useful, even it is limited to a 2-dimensional case, which provides simple but intuitional knowledge for further analysis in many cases.

In chapter 6, we analyzed the characteristics of W/B weight restrictions theoretically and compared with those of C/R weight restriction empirically. We showed that under W/B weight restriction, each DMU takes all different weight vectors and some DMUs may have limiting efficiency score.

To see the practical difference between two restriction methods, we performed an empirical study and showed that first, even though C/R and W/B weight restriction take different weight vectors for each DMU, ranking results of empirical study are appeared to be very similar in many cases of multiple-input, multiple-output situation, second, however the ranking result may be far different when a certain DMU has limiting efficiency score under W/B weight restriction. This is based on the fact that while C/R weight restriction allows flexible substitution among inputs or outputs, W/B weight restriction would not allow this flexible substitution.

In chapter 7, we presented alternative models, which can measure each of overall efficiency (OE) with cone-ratio weight restrictions and compared with previous models using examples. The only difference between the proposed and CCR model is the added cost (price) vector constraints, which results in the DEA models with cone-ratio weight restrictions.

That represents that the proposed model can directly measure the overall efficiency score of DMU  $j$  and previous models would have the same characteristics with those of cone-ratio weight restriction models. For example, 1) we can apply the models when we know just the possible ranges of costs or prices. 2) Also they allow flexible substitution for measuring



overall efficiency.

## **8.2 Future Research**

Although many kinds of DEA ranking models have been developed, it is still necessary to develop more precise weighting schemes for DEA ranking models.

The possible future research areas on DEA ranking models include the following. First, it would be useful to extend DEA ranking models to consider the variable returns to scales, since most of previous ranking models have the assumption of constant returns to scale. Second, more importantly, the linearity assumption in DEA ranking model may be problematic when the preference of a decision maker cannot be represented as a linear function. Third, in case that the decision maker has the preference, which is linear but varies according to certain ranges of input or output quantity, it is not easy to reflect to DEA ranking models.

While models for measuring overall efficiency can give some useful information, there still needs to be further research to overcome the followings. First, these models cannot tell the specific way to increase overall efficiency except telling optimal ratios of costs (prices). Second, the overall efficiency of DMU  $j$  is calculated based on the assumption that all the other DMUs are assumed to use DMU  $j$ 's cost or price vectors, and it may not appropriately reflect a variety of management strategies.

Finally, even though we presented the characteristics of W/B weight restriction and compared with C/R weight restriction, further research is still necessary to thoroughly analyze the merits of setting restrictions on the virtual inputs and outputs.

## APPENDIX . RESULTS OF EMPIRICAL STUDY

### A.1 Application data

#### (a) FMS selection

#### (b) Car Selection

DMU	FMS Data						DMU	Car Selection Data					
	$x_1$	$x_2$	$y_1$	$y_2$	$y_3$	$y_4$		$x_1$	$x_2$	$x_3$	$x_4$	$y_1$	$y_2$
1	17.02	5.0	42	45.3	14.2	30.1	1	28.28	2	6.8	5.4	29	115
2	16.46	4.5	39	40.1	13.0	29.8	2	44.30	5	6.3	4.8	50	140
3	11.76	6.0	26	39.6	13.8	24.5	3	44.80	4	5.8	4.3	50	142
4	10.52	4.0	22	36.0	11.3	25.0	4	29.80	4	6.3	5.6	34	110
5	9.50	3.8	21	34.2	12.0	20.4	5	44.90	4	6.3	4.5	45	137
6	4.79	5.4	10	20.1	5.0	16.5	6	42.95	4	5.8	4.6	50	143
7	6.21	6.2	14	26.5	7.0	19.7	7	33.30	4	8.8	6.0	53	145
8	11.12	6.0	25	35.9	9.0	24.7	8	38.01	5	8.7	5.1	55	152
9	3.67	8.0	4	17.4	0.1	18.1	9	43.40	4	9.2	6.1	50	146
10	8.93	7.0	16	34.3	6.5	20.6	10	47.50	6	9.0	5.5	73	163
11	17.74	7.1	43	45.6	14.0	31.1	11	21.55	3	7.3	6.1	24	105
12	14.85	6.2	27	38.7	13.8	25.4	12	34.30	4	7.1	4.8	45	140
							13	37.80	5	7.8	5.8	48	137
							14	36.40	5	7.2	5.5	53	145
							15	44.60	5	8.0	5.2	60	145
							16	44.90	5	8.4	5.6	55	160
							17	46.30	6	7.6	5.9	70	165
							18	45.30	6	8.8	5.6	65	155
							19	26.15	7	8.3	6.4	58	139
							20	30.61	7	9.5	7.5	65	147
							21	251.00	11	12.0	9.0	200	220
							22	263.24	20	15.0	11.0	218	229
							23	362.80	34	17.0	12.0	231	220
							24	231.80	16	13.0	8.5	204	235
							25	267.00	31	18.0	11.0	295	245
							26	239.00	11	14.0	8.0	162	220
							27	322.00	17	16.0	9.0	214	240
							28	325.00	20	17.0	10.0	255	250

(c) Location of electrical power station (d) Location of solid waste management system

DMU	Hydro electrical power station						DMU	Solid waste management system							
	$x_1$	$x_2$	$x_3$	$x_4$	$y_1$	$y_2$		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$y_1$	$y_2$	$y_3$
1	80	6	5.4	8	90	5	1	656	552678100	609	1190	670	5	14	13900
2	65	2	9.7	1	58	1	2	786	539113200	575	1190	682	4	18	23600
3	83	4	7.2	4	60	7	3	912	480565400	670	1222	594	4	24	39767
4	40	10	7.5	7	80	10	4	589	559780715	411	1191	443	9	10	13900
5	52	6	2	3	72	8	5	706	532286214	325	1191	404	7	14	23600
6	94	7	3.6	5	96	6	6	834	470613514	500	1226	384	6.5	18	40667
							7	580	560987877	398	1191	420	9	10	13900
							8	682	532224858	314	1191	393	7	14	23600
							9	838	466586058	501	1229	373	6.5	22	41747
							10	579	561555877	373	1191	405	9	9	13900
							11	688	532302258	292	1191	370	7	13	23600
							12	838	465356158	499	1230	361	6.5	17	42467
							13	595	560500215	500	1191	538	9	12	13900
							14	709	532974014	402	1191	489	7	17	23600
							15	849	474137314	648	1226	538	6.5	20	40667
							16	604	560500215	500	1191	538	9	12	13900
							17	736	532974014	402	1191	489	7	17	23600
							18	871	474137314	648	1226	538	6.5	20	40667
							19	579	568674539	495	1193	558	9	7	13900
							20	695	536936873	424	1195	535	6	18	23600
							21	827	457184239	651	1237	513	7	16	45167
							22	982	457206173	651	1239	513	7	16	45167

(e) Economic performance of Chinese cities (f) Evaluating regions in Serbia

DMU	Economic performance of Cities				
	$x_1$	$x_2$	$y_1$	$y_2$	$y_3$
1	2874.8	16738	160.89	80800	5092
2	946.3	691	21.14	18172	6563
3	6854	43024	375.25	144530	2437
4	2305.1	10815	176.68	70318	3145
5	1010.3	2099	102.12	55419	1225
6	282.3	757	59.17	27422	246
7	17478.6	116900	1029.09	351390	14604
8	661.8	2024	30.07	23550	1126
9	1544.2	3218	160.58	59406	2230
10	428.4	574	53.69	47504	430
11	6228.1	29842	258.09	151356	4649
12	697.7	3394	38.02	45336	1555
13	106.4	367	7.07	8236	121
14	4539.3	45809	116.46	56135	956
15	957.8	16947	29.2	17554	231
16	1209.2	15741	65.36	62341	618
17	972.4	23822	54.52	25230	513
18	2192	10943	25.24	40267	895

DMU	Evaluating regions in Serbia							
	$x_1$	$x_2$	$x_3$	$x_4$	$y_1$	$y_2$	$y_3$	$y_4$
1	1781	1621	436	205	174	497	22	66
2	3257	2718	314	221	172	497	22	61
3	2328	2045	475	179	148	417	18	53
4	4248	3228	419	328	142	760	35	84
5	2406	1523	345	215	160	443	22	62
6	4018	5514	1314	553	487	1925	63	192
7	3480	1941	507	309	220	521	36	79
8	3279	1496	321	339	109	699	38	71
9	2475	932	158	200	37	431	19	48
10	1244	2445	656	226	83	516	24	52
11	3855	1594	429	253	82	487	20	39
12	2387	4029	658	312	109	828	35	95
13	2614	1662	625	264	112	632	24	62
14	3506	3080	282	178	127	461	16	47
15	3625	820	254	158	71	522	13	43
16	6141	2018	524	335	180	782	39	95
17	3016	1332	489	230	168	518	25	72
18	3917	1756	484	300	100	714	37	70
19	2957	1321	532	296	169	652	29	75
20	2440	2011	945	382	206	1583	38	112
21	2230	506	151	111	35	267	11	23
22	2764	453	199	116	43	266	10	33
23	2771	682	294	255	100	640	26	60
24	3519	782	341	243	110	546	29	55
25	3120	1441	578	672	85	976	13	62
26	2420	1070	303	414	40	473	6	31
27	1909	644	395	376	35	289	9	27
28	2051	517	150	275	23	304	6	23
29	1412	235	222	217	28	164	7	17
30	3222	15123	5321	1602	1284	6781	165	580

**A.2 (FMS Data) Results comparison of SI, A-P and cross-efficiency scores**

<b>DMU</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10</b>	<b>11</b>	<b>12</b>
<b>1</b>	<b>1.000</b>	<b>0.915</b>	<b>1.000</b>	<b>0.836</b>	<b>0.714</b>	<b>0.607</b>	<b>0.943</b>	<b>1.000</b>	<b>0.359</b>	<b>0.764</b>	<b>1.000</b>	<b>1.000</b>
<b>2</b>	<b>0.956</b>	<b>1.000</b>	<b>0.959</b>	<b>0.881</b>	<b>0.686</b>	<b>0.601</b>	<b>0.897</b>	<b>0.977</b>	<b>0.367</b>	<b>0.704</b>	<b>0.966</b>	<b>0.984</b>
<b>3</b>	<b>0.919</b>	<b>0.606</b>	<b>0.982</b>	<b>0.752</b>	<b>0.862</b>	<b>0.774</b>	<b>0.925</b>	<b>0.949</b>	<b>0.422</b>	<b>0.904</b>	<b>0.924</b>	<b>0.953</b>
<b>4</b>	<b>0.910</b>	<b>0.915</b>	<b>0.927</b>	<b>1.000</b>	<b>0.862</b>	<b>0.801</b>	<b>0.891</b>	<b>1.000</b>	<b>0.482</b>	<b>0.956</b>	<b>0.896</b>	<b>1.000</b>
<b>5</b>	<b>0.956</b>	<b>0.794</b>	<b>1.000</b>	<b>0.881</b>	<b>1.000</b>	<b>0.815</b>	<b>0.943</b>	<b>1.000</b>	<b>0.435</b>	<b>1.000</b>	<b>0.927</b>	<b>1.000</b>
<b>6</b>	<b>0.825</b>	<b>0.438</b>	<b>0.932</b>	<b>0.744</b>	<b>0.547</b>	<b>1.000</b>	<b>0.943</b>	<b>0.962</b>	<b>0.698</b>	<b>0.951</b>	<b>0.954</b>	<b>0.967</b>
<b>7</b>	<b>0.897</b>	<b>0.459</b>	<b>1.000</b>	<b>0.748</b>	<b>0.628</b>	<b>0.973</b>	<b>1.000</b>	<b>1.000</b>	<b>0.643</b>	<b>1.000</b>	<b>1.000</b>	<b>1.000</b>
<b>8</b>	<b>0.916</b>	<b>0.608</b>	<b>0.927</b>	<b>0.777</b>	<b>0.584</b>	<b>0.687</b>	<b>0.917</b>	<b>0.961</b>	<b>0.450</b>	<b>0.860</b>	<b>0.945</b>	<b>0.950</b>
<b>9</b>	<b>0.482</b>	<b>0.315</b>	<b>0.422</b>	<b>0.632</b>	<b>0.010</b>	<b>0.973</b>	<b>0.659</b>	<b>0.753</b>	<b>1.000</b>	<b>0.849</b>	<b>0.672</b>	<b>0.724</b>
<b>10</b>	<b>0.772</b>	<b>0.427</b>	<b>0.760</b>	<b>0.642</b>	<b>0.454</b>	<b>0.679</b>	<b>0.822</b>	<b>0.833</b>	<b>0.468</b>	<b>0.954</b>	<b>0.783</b>	<b>0.795</b>
<b>11</b>	<b>0.956</b>	<b>0.665</b>	<b>0.977</b>	<b>0.719</b>	<b>0.625</b>	<b>0.590</b>	<b>0.923</b>	<b>0.951</b>	<b>0.355</b>	<b>0.714</b>	<b>0.983</b>	<b>0.953</b>
<b>12</b>	<b>0.760</b>	<b>0.608</b>	<b>0.801</b>	<b>0.687</b>	<b>0.727</b>	<b>0.625</b>	<b>0.748</b>	<b>0.794</b>	<b>0.347</b>	<b>0.720</b>	<b>0.759</b>	<b>0.801</b>

<b>SI</b>	<b>0.862</b>	<b>0.646</b>	<b>0.891</b>	<b>0.775</b>	<b>0.642</b>	<b>0.760</b>	<b>0.884</b>	<b>0.932</b>	<b>0.502</b>	<b>0.865</b>	<b>0.901</b>	<b>0.927</b>
<b>rank</b>	<b>6</b>	<b>3</b>	<b>9</b>	<b>5</b>	<b>2</b>	<b>4</b>	<b>8</b>	<b>11</b>	<b>1</b>	<b>7</b>	<b>10</b>	<b>11</b>

<b>CE</b>	<b>0.848</b>	<b>0.839</b>	<b>0.777</b>	<b>0.844</b>	<b>0.867</b>	<b>0.727</b>	<b>0.758</b>	<b>0.724</b>	<b>0.564</b>	<b>0.618</b>	<b>0.747</b>	<b>0.667</b>
<b>rank</b>	<b>2</b>	<b>4</b>	<b>5</b>	<b>3</b>	<b>1</b>	<b>8</b>	<b>6</b>	<b>9</b>	<b>12</b>	<b>11</b>	<b>7</b>	<b>10</b>

<b>A-P</b>	<b>1.045</b>	<b>1.093</b>	<b>0.982</b>	<b>1.134</b>	<b>1.160</b>	<b>1.028</b>	<b>1.06</b>	<b>0.961</b>	<b>1.432</b>	<b>0.953</b>	<b>0.983</b>	<b>0.801</b>
<b>rank</b>	<b>6</b>	<b>4</b>	<b>8</b>	<b>3</b>	<b>2</b>	<b>7</b>	<b>5</b>	<b>10</b>	<b>1</b>	<b>11</b>	<b>8</b>	<b>12</b>

**A.3 (Car Selection Data) Results comparison of SI, A-P and cross-efficiency scores**

DMU	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1.000	0.811	0.629	0.926	0.917	0.888	0.895	0.779	1.000	0.625	0.924	0.935	0.934	0.934	0.917
2	0.487	<b>0.940</b>	0.884	0.925	0.869	0.923	0.734	0.872	0.838	0.866	0.673	0.876	0.916	0.916	0.856
3	0.617	1.000	<b>1.000</b>	0.969	1.000	0.972	0.804	0.920	1.000	0.941	0.700	0.934	0.971	0.971	1.000
4	0.478	0.784	0.589	<b>0.892</b>	0.721	0.867	0.714	0.742	0.708	0.622	0.763	0.831	0.875	0.875	0.707
5	0.597	0.925	0.918	0.905	<b>0.946</b>	0.902	0.738	0.857	0.918	0.854	0.675	0.883	0.900	0.900	0.912
6	0.622	1.000	0.941	1.000	0.983	<b>1.000</b>	0.827	0.919	0.994	0.912	0.731	0.952	1.000	1.000	0.993
7	0.630	0.925	0.734	0.933	0.900	0.892	<b>1.000</b>	0.931	1.000	0.818	0.918	0.952	0.975	0.975	1.000
8	0.529	0.993	0.905	0.927	0.940	0.895	0.873	<b>1.000</b>	0.922	0.941	0.830	0.952	0.946	0.946	0.947
9	0.635	0.827	0.723	0.817	0.863	0.792	0.826	0.811	<b>0.936</b>	0.747	0.739	0.838	0.845	0.845	0.919
10	0.472	0.982	0.918	0.878	0.879	0.859	0.895	0.973	0.958	<b>1.000</b>	0.717	0.879	0.926	0.926	1.000
11	0.609	0.729	0.506	0.895	0.731	0.843	0.760	0.714	0.689	0.518	<b>1.000</b>	0.878	0.874	0.874	0.658
12	0.609	1.000	0.877	1.000	0.979	0.972	0.867	0.973	0.943	0.883	0.865	<b>1.000</b>	1.000	1.000	0.947
13	0.476	0.867	0.715	0.887	0.795	0.863	0.771	0.839	0.793	0.758	0.751	0.860	<b>0.890</b>	0.890	0.807
14	0.504	0.975	0.801	0.995	0.869	0.972	0.858	0.939	0.874	0.860	0.820	0.952	1.000	<b>1.000</b>	0.897
15	0.504	0.924	0.857	0.856	0.866	0.841	0.844	0.900	0.936	0.907	0.694	0.853	0.893	0.893	<b>0.960</b>
16	0.556	0.945	0.863	0.920	0.920	0.900	0.810	0.913	0.906	0.867	0.761	0.916	0.927	0.927	0.915
17	0.478	1.000	0.861	0.976	0.864	0.965	0.881	0.951	0.926	0.941	0.741	0.920	1.000	1.000	0.962
18	0.449	0.937	0.852	0.864	0.835	0.844	0.830	0.924	0.874	0.919	0.709	0.857	0.895	0.895	0.911
19	0.345	0.957	0.668	1.000	0.704	0.952	0.857	0.973	0.709	0.822	0.924	0.930	1.000	1.000	0.760
20	0.365	0.879	0.606	0.921	0.676	0.876	0.895	0.892	0.747	0.765	0.875	0.866	0.951	0.951	0.791
21	0.348	0.529	0.839	0.365	0.513	0.387	0.593	0.470	1.000	0.791	0.208	0.347	0.475	0.475	1.000
22	0.199	0.507	0.721	0.345	0.401	0.364	0.521	0.460	0.704	0.752	0.196	0.320	0.444	0.444	0.751
23	0.113	0.393	0.648	0.250	0.271	0.266	0.365	0.353	0.465	0.625	0.133	0.225	0.324	0.324	0.513
24	0.255	0.589	0.941	0.405	0.504	0.427	0.579	0.531	0.828	0.876	0.231	0.381	0.506	0.506	0.873
25	0.137	0.576	0.812	0.346	0.350	0.362	0.570	0.544	0.656	0.941	0.195	0.315	0.474	0.474	0.738
26	0.348	0.514	0.909	0.366	0.543	0.384	0.514	0.460	0.874	0.746	0.217	0.358	0.445	0.445	0.873
27	0.245	0.466	0.912	0.307	0.441	0.325	0.472	0.416	0.767	0.738	0.174	0.295	0.393	0.393	0.790
28	0.217	0.493	0.878	0.312	0.420	0.330	0.524	0.448	0.788	0.803	0.177	0.298	0.419	0.419	0.829

<b>SI</b>	<b>0.458</b>	<b>0.802</b>	<b>0.804</b>	<b>0.756</b>	<b>0.739</b>	<b>0.745</b>	<b>0.744</b>	<b>0.768</b>	<b>0.848</b>	<b>0.816</b>	<b>0.619</b>	<b>0.736</b>	<b>0.793</b>	<b>0.793</b>	<b>0.865</b>
rank	1	19	20	14	11	13	12	15	27	22	5	10	18	17	28

<b>CE</b>	<b>0.770</b>	<b>0.760</b>	<b>0.833</b>	<b>0.670</b>	<b>0.767</b>	<b>0.837</b>	<b>0.824</b>	<b>0.820</b>	<b>0.731</b>	<b>0.816</b>	<b>0.672</b>	<b>0.834</b>	<b>0.723</b>	<b>0.808</b>	<b>0.777</b>
rank	13	15	3	20	14	1	5	6	17	7	19	2	18	8	11

<b>A-P</b>	<b>1.575</b>	<b>0.940</b>	<b>1.063</b>	<b>0.892</b>	<b>0.948</b>	<b>1.029</b>	<b>1.117</b>	<b>1.028</b>	<b>0.936</b>	<b>1.062</b>	<b>1.083</b>	<b>1.050</b>	<b>0.890</b>	<b>1.000</b>	<b>0.960</b>
rank	1	23	10	26	22	14	4	15	24	9	7	11	27	16	19

**A.3 (Continued) (Car Selection Data) Results comparison of SI, A-P and cross-  
efficiency scores**

<b>DMU</b>	<b>16</b>	<b>17</b>	<b>18</b>	<b>19</b>	<b>20</b>	<b>21</b>	<b>22</b>	<b>23</b>	<b>24</b>	<b>25</b>	<b>26</b>	<b>27</b>	<b>28</b>
<b>1</b>	0.900	0.646	0.785	0.829	0.717	0.466	0.604	0.255	0.578	0.394	0.659	0.482	0.281
<b>2</b>	0.894	0.855	0.910	0.739	0.699	0.523	0.874	0.475	0.838	0.635	0.870	0.695	0.482
<b>3</b>	1.000	0.904	0.959	0.757	0.738	0.619	0.945	0.518	0.936	0.675	1.000	0.795	0.547
<b>4</b>	0.754	0.710	0.768	0.756	0.668	0.407	0.671	0.319	0.572	0.444	0.598	0.459	0.297
<b>5</b>	0.927	0.795	0.885	0.725	0.663	0.538	0.822	0.430	0.842	0.591	0.913	0.714	0.476
<b>6</b>	0.996	0.922	0.962	0.780	0.762	0.619	0.954	0.515	0.898	0.660	0.949	0.752	0.516
<b>7</b>	0.995	0.816	0.941	0.903	0.966	0.543	0.697	0.364	0.684	0.638	0.774	0.617	0.429
<b>8</b>	1.000	0.816	1.000	0.894	0.852	0.504	0.727	0.387	0.799	0.711	0.901	0.722	0.503
<b>9</b>	0.900	0.705	0.821	0.744	0.756	0.501	0.639	0.329	0.658	0.549	0.754	0.593	0.400
<b>10</b>	1.000	0.922	1.000	0.792	0.918	0.594	0.826	0.495	0.853	0.831	0.936	0.789	0.601
<b>11</b>	0.731	0.560	0.714	0.869	0.644	0.311	0.507	0.195	0.466	0.316	0.511	0.367	0.204
<b>12</b>	1.000	0.823	0.984	0.903	0.804	0.511	0.770	0.384	0.794	0.627	0.877	0.682	0.450
<b>13</b>	0.848	0.772	0.861	0.784	0.747	0.461	0.713	0.369	0.676	0.570	0.723	0.573	0.393
<b>14</b>	0.942	0.896	0.970	0.865	0.845	0.526	0.832	0.441	0.768	0.661	0.809	0.648	0.453
<b>15</b>	0.952	0.857	0.927	0.749	0.835	0.571	0.789	0.456	0.799	0.724	0.879	0.725	0.534
<b>16</b>	<b>0.949</b>	0.803	0.930	0.809	0.761	0.512	0.760	0.397	0.786	0.632	0.865	0.683	0.465
<b>17</b>	0.973	<b>1.000</b>	1.000	0.806	0.895	0.609	0.947	0.551	0.850	0.769	0.880	0.731	0.545
<b>18</b>	0.932	0.861	<b>0.948</b>	0.779	0.844	0.534	0.778	0.449	0.792	0.744	0.860	0.713	0.529
<b>19</b>	0.848	0.922	1.000	<b>1.000</b>	0.966	0.443	0.767	0.418	0.662	0.707	0.671	0.556	0.412
<b>20</b>	0.835	0.894	0.924	0.903	<b>1.000</b>	0.472	0.729	0.409	0.611	0.676	0.633	0.531	0.401
<b>21</b>	0.690	0.816	0.543	0.235	0.607	<b>1.000</b>	1.000	1.000	0.936	0.758	1.000	1.000	0.959
<b>22</b>	0.587	0.777	0.530	0.231	0.570	0.670	<b>0.905</b>	0.874	0.818	0.746	0.795	0.811	0.795
<b>23</b>	0.420	0.617	0.412	0.164	0.416	0.457	0.778	<b>0.820</b>	0.737	0.622	0.640	0.677	0.690
<b>24</b>	0.684	0.845	0.605	0.272	0.619	0.767	1.000	0.953	<b>1.000</b>	0.831	1.000	1.000	0.959
<b>25</b>	0.615	0.922	0.632	0.244	0.678	0.621	1.000	1.000	0.936	<b>1.000</b>	0.838	0.913	0.959
<b>26</b>	0.642	0.671	0.514	0.248	0.513	0.767	0.787	0.709	0.884	0.660	<b>0.998</b>	0.952	0.864
<b>27</b>	0.578	0.669	0.475	0.203	0.492	0.726	0.817	0.821	0.917	0.683	0.976	<b>0.982</b>	0.948
<b>28</b>	0.611	0.758	0.517	0.209	0.565	0.757	0.894	0.918	0.936	0.784	0.962	1.000	<b>1.000</b>
<b>SI</b>	<b>0.829</b>	<b>0.806</b>	<b>0.804</b>	<b>0.650</b>	<b>0.734</b>	<b>0.572</b>	<b>0.805</b>	<b>0.545</b>	<b>0.787</b>	<b>0.666</b>	<b>0.831</b>	<b>0.720</b>	<b>0.575</b>
rank	25	23	21	26	9	3	22	2	16	7	26	8	4
<b>CE</b>	<b>0.785</b>	<b>0.827</b>	<b>0.771</b>	<b>0.787</b>	<b>0.755</b>	<b>0.615</b>	<b>0.520</b>	<b>0.404</b>	<b>0.609</b>	<b>0.564</b>	<b>0.561</b>	<b>0.531</b>	<b>0.556</b>
rank	10	4	12	9	16	21	27	28	22	23	24	26	25
<b>A-P</b>	<b>0.949</b>	<b>1.084</b>	<b>0.948</b>	<b>1.107</b>	<b>1.035</b>	<b>1.304</b>	<b>0.905</b>	<b>0.820</b>	<b>1.068</b>	<b>1.204</b>	<b>0.998</b>	<b>0.982</b>	<b>1.043</b>
rank	20	6	21	5	13	2	25	28	8	3	17	18	12

**A.4 (FMS Data) Random weight ratios are used (N = 20)**

<b>N</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10</b>	<b>11</b>	<b>12</b>
1	0.956	0.934	0.934	0.988	1.000	0.905	0.948	0.910	0.648	0.787	0.899	0.783
2	1.000	0.975	0.821	0.895	0.928	0.561	0.635	0.782	0.255	0.587	0.879	0.721
3	1.000	0.976	0.808	0.846	0.879	0.573	0.647	0.791	0.236	0.570	0.907	0.704
4	0.988	0.960	0.839	0.985	1.000	0.587	0.656	0.795	0.345	0.657	0.832	0.743
5	0.686	0.670	0.764	0.827	0.810	0.954	0.946	0.768	1.000	0.776	0.661	0.623
6	0.909	0.881	0.946	0.955	1.000	0.931	0.976	0.877	0.595	0.765	0.867	0.780
7	0.899	0.867	0.865	0.982	1.000	0.699	0.760	0.813	0.460	0.727	0.789	0.741
8	0.948	0.930	0.855	0.986	1.000	0.638	0.699	0.793	0.373	0.651	0.817	0.751
9	0.867	0.855	0.873	1.000	0.987	0.813	0.849	0.834	0.607	0.746	0.780	0.747
10	0.809	0.769	0.871	0.954	1.000	0.731	0.793	0.777	0.464	0.736	0.721	0.729
11	0.961	0.946	0.890	0.996	1.000	0.748	0.801	0.852	0.483	0.708	0.862	0.769
12	0.874	0.839	0.919	0.992	1.000	0.953	0.989	0.904	0.808	0.880	0.818	0.757
13	0.935	0.908	0.903	0.982	1.000	0.795	0.850	0.864	0.529	0.754	0.854	0.765
14	0.895	0.865	0.917	0.973	1.000	0.863	0.913	0.867	0.603	0.785	0.832	0.763
15	0.973	0.963	0.912	1.000	0.992	0.846	0.886	0.892	0.589	0.736	0.898	0.781
16	1.000	0.985	0.763	0.933	0.956	0.443	0.510	0.691	0.211	0.522	0.786	0.703
17	0.773	0.735	0.915	0.939	1.000	0.902	0.948	0.800	0.611	0.788	0.724	0.741
18	0.918	0.892	0.876	0.987	1.000	0.727	0.784	0.831	0.480	0.728	0.816	0.752
19	0.906	0.885	0.825	1.000	0.989	0.644	0.698	0.790	0.447	0.694	0.765	0.726
20	0.844	0.833	0.886	0.974	1.000	0.766	0.812	0.780	0.457	0.664	0.761	0.756
<b>mean</b>	<b>0.907</b>	<b>0.883</b>	<b>0.869</b>	<b>0.960</b>	<b>0.977</b>	<b>0.754</b>	<b>0.805</b>	<b>0.821</b>	<b>0.510</b>	<b>0.713</b>	<b>0.813</b>	<b>0.742</b>
<b>rank</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>2</b>	<b>1</b>	<b>9</b>	<b>8</b>	<b>6</b>	<b>12</b>	<b>11</b>	<b>7</b>	<b>10</b>
<b>CE</b>	<b>0.848</b>	<b>0.839</b>	<b>0.777</b>	<b>0.844</b>	<b>0.867</b>	<b>0.727</b>	<b>0.758</b>	<b>0.724</b>	<b>0.564</b>	<b>0.618</b>	<b>0.747</b>	<b>0.667</b>
<b>rank</b>	<b>2</b>	<b>4</b>	<b>5</b>	<b>3</b>	<b>1</b>	<b>8</b>	<b>6</b>	<b>9</b>	<b>12</b>	<b>11</b>	<b>7</b>	<b>10</b>
<b>Equal</b>	<b>0.907</b>	<b>0.883</b>	<b>0.888</b>	<b>0.986</b>	<b>1.000</b>	<b>0.769</b>	<b>0.822</b>	<b>0.839</b>	<b>0.515</b>	<b>0.738</b>	<b>0.817</b>	<b>0.757</b>
<b>rank</b>	<b>3</b>	<b>5</b>	<b>4</b>	<b>2</b>	<b>1</b>	<b>9</b>	<b>7</b>	<b>6</b>	<b>12</b>	<b>11</b>	<b>8</b>	<b>10</b>



**A.5 (Car Selection Data) Random weight ratios are used (N = 30)**

<b>N</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10</b>	<b>11</b>	<b>12</b>
1	0.780	0.892	1.000	0.711	0.900	0.995	0.919	0.926	0.833	0.998	0.613	0.924
2	0.641	0.876	1.000	0.625	0.880	0.977	0.786	0.833	0.738	0.946	0.483	0.830
3	0.763	0.908	0.966	0.758	0.880	0.971	0.902	0.945	0.796	0.973	0.679	0.947
4	0.897	0.865	1.000	0.719	0.933	0.989	0.855	0.875	0.824	0.839	0.708	0.936
5	0.914	0.870	0.942	0.798	0.874	0.957	1.000	0.971	0.868	0.957	0.808	0.986
6	0.720	0.860	0.947	0.686	0.849	0.945	0.906	0.916	0.801	1.000	0.579	0.892
7	0.885	0.897	1.000	0.756	0.928	0.996	0.949	0.971	0.868	0.956	0.733	0.988
8	0.828	0.845	0.940	0.716	0.855	0.943	0.982	0.948	0.861	1.000	0.660	0.931
9	0.684	0.909	0.999	0.705	0.881	1.000	0.794	0.824	0.728	0.914	0.561	0.853
10	0.639	0.864	0.966	0.651	0.841	0.963	0.825	0.832	0.743	0.971	0.487	0.822
11	0.988	0.849	0.931	0.815	0.866	0.954	1.000	0.930	0.871	0.908	0.862	0.972
12	0.858	0.895	0.982	0.776	0.894	0.993	0.962	0.942	0.851	0.977	0.720	0.959
13	0.807	0.892	0.988	0.737	0.895	0.989	0.943	0.944	0.843	0.999	0.656	0.943
14	0.705	0.804	0.890	0.657	0.792	0.893	0.931	0.896	0.802	1.000	0.547	0.855
15	0.823	0.904	1.000	0.749	0.917	0.999	0.878	0.909	0.809	0.916	0.691	0.945
16	0.874	0.880	0.951	0.786	0.875	0.963	1.000	0.981	0.865	0.990	0.765	0.981
17	0.873	0.860	0.927	0.775	0.857	0.939	1.000	0.980	0.860	0.977	0.771	0.976
18	0.623	0.870	0.928	0.685	0.813	0.937	0.794	0.815	0.699	0.930	0.525	0.819
19	0.731	0.846	0.950	0.692	0.835	0.960	0.896	0.847	0.791	0.971	0.553	0.847
20	0.644	0.729	0.748	0.661	0.683	0.764	0.853	0.854	0.693	0.879	0.619	0.813
21	0.784	0.803	0.848	0.743	0.772	0.869	0.972	0.921	0.801	0.959	0.713	0.899
22	0.725	0.876	1.000	0.666	0.901	0.984	0.767	0.812	0.736	0.852	0.572	0.854
23	0.730	0.902	1.000	0.701	0.901	0.991	0.830	0.878	0.768	0.930	0.598	0.897
24	0.715	0.871	1.000	0.654	0.897	0.981	0.787	0.828	0.751	0.887	0.551	0.854
25	0.658	0.839	0.936	0.654	0.818	0.937	0.871	0.851	0.767	0.992	0.497	0.828
26	0.772	0.922	0.987	0.756	0.902	0.987	0.903	0.960	0.807	0.981	0.678	0.961
27	0.812	0.881	1.000	0.712	0.908	0.994	0.886	0.895	0.821	0.936	0.638	0.918
28	0.769	0.729	0.758	0.716	0.700	0.781	0.940	0.879	0.753	0.876	0.750	0.856
29	0.869	0.898	0.993	0.771	0.908	1.000	0.959	0.946	0.858	0.973	0.720	0.965
30	0.688	0.880	0.953	0.691	0.853	0.950	0.876	0.915	0.776	0.999	0.572	0.892
<b>mean</b>	<b>0.773</b>	<b>0.864</b>	<b>0.951</b>	<b>0.717</b>	<b>0.860</b>	<b>0.953</b>	<b>0.899</b>	<b>0.901</b>	<b>0.799</b>	<b>0.950</b>	<b>0.644</b>	<b>0.905</b>
<b>rank</b>	<b>18</b>	<b>12</b>	<b>3</b>	<b>19</b>	<b>13</b>	<b>2</b>	<b>8</b>	<b>7</b>	<b>16</b>	<b>4</b>	<b>20</b>	<b>5</b>
<b>CE</b>	<b>0.813</b>	<b>0.812</b>	<b>0.900</b>	<b>0.703</b>	<b>0.823</b>	<b>0.902</b>	<b>0.861</b>	<b>0.847</b>	<b>0.774</b>	<b>0.859</b>	<b>0.678</b>	<b>0.868</b>
<b>rank</b>	<b>12</b>	<b>13</b>	<b>2</b>	<b>19</b>	<b>10</b>	<b>1</b>	<b>5</b>	<b>7</b>	<b>16</b>	<b>6</b>	<b>20</b>	<b>4</b>
<b>Equal</b>	<b>0.816</b>	<b>0.901</b>	<b>0.998</b>	<b>0.727</b>	<b>0.905</b>	<b>1.000</b>	<b>0.933</b>	<b>0.936</b>	<b>0.839</b>	<b>0.986</b>	<b>0.613</b>	<b>0.946</b>
<b>rank</b>	<b>18</b>	<b>13</b>	<b>3</b>	<b>19</b>	<b>12</b>	<b>1</b>	<b>9</b>	<b>6</b>	<b>16</b>	<b>4</b>	<b>20</b>	<b>5</b>

**A.5 (Continued) Random weight ratios are used (N = 30)**

<b>N</b>	<b>13</b>	<b>14</b>	<b>15</b>	<b>16</b>	<b>17</b>	<b>18</b>	<b>19</b>	<b>20</b>
1	0.809	0.914	0.942	0.898	0.996	0.919	0.825	0.822
2	0.731	0.836	0.891	0.830	0.956	0.862	0.722	0.713
3	0.839	0.951	0.910	0.903	1.000	0.920	0.951	0.891
4	0.773	0.851	0.836	0.885	0.859	0.799	0.679	0.657
5	0.859	0.956	0.910	0.920	0.966	0.906	0.930	0.908
6	0.793	0.900	0.928	0.870	0.987	0.917	0.857	0.850
7	0.837	0.929	0.918	0.935	0.951	0.901	0.826	0.799
8	0.814	0.913	0.938	0.893	0.973	0.919	0.857	0.868
9	0.771	0.887	0.869	0.837	0.991	0.854	0.807	0.779
10	0.751	0.866	0.906	0.822	1.000	0.883	0.798	0.803
11	0.847	0.938	0.880	0.901	0.937	0.864	0.881	0.878
12	0.845	0.951	0.929	0.913	1.000	0.915	0.894	0.883
13	0.828	0.933	0.941	0.908	1.000	0.926	0.866	0.857
14	0.769	0.874	0.921	0.836	0.968	0.906	0.851	0.873
15	0.814	0.913	0.883	0.900	0.949	0.868	0.812	0.776
16	0.861	0.964	0.932	0.924	0.992	0.930	0.951	0.929
17	0.852	0.951	0.918	0.914	0.970	0.919	0.948	0.924
18	0.761	0.884	0.864	0.806	1.000	0.864	0.880	0.849
19	0.775	0.886	0.915	0.835	1.000	0.885	0.818	0.847
20	0.740	0.841	0.795	0.765	0.872	0.827	1.000	0.940
21	0.816	0.921	0.885	0.848	0.960	0.896	1.000	0.988
22	0.734	0.828	0.829	0.832	0.895	0.799	0.683	0.658
23	0.783	0.888	0.884	0.871	0.961	0.870	0.792	0.759
24	0.736	0.831	0.855	0.837	0.911	0.823	0.692	0.676
25	0.759	0.872	0.920	0.824	1.000	0.897	0.823	0.843
26	0.842	0.952	0.919	0.918	1.000	0.927	0.929	0.869
27	0.791	0.888	0.900	0.886	0.948	0.870	0.767	0.759
28	0.773	0.866	0.807	0.791	0.872	0.828	1.000	0.974
29	0.842	0.945	0.929	0.919	0.990	0.911	0.868	0.856
30	0.797	0.909	0.922	0.870	1.000	0.922	0.888	0.859
<b>mean</b>	<b>0.798</b>	<b>0.901</b>	<b>0.896</b>	<b>0.870</b>	<b>0.964</b>	<b>0.884</b>	<b>0.853</b>	<b>0.836</b>
<b>rank</b>	<b>17</b>	<b>6</b>	<b>9</b>	<b>11</b>	<b>1</b>	<b>10</b>	<b>14</b>	<b>15</b>
<b>CE</b>	<b>0.757</b>	<b>0.846</b>	<b>0.824</b>	<b>0.820</b>	<b>0.877</b>	<b>0.796</b>	<b>0.794</b>	<b>0.768</b>
<b>rank</b>	<b>18</b>	<b>8</b>	<b>9</b>	<b>11</b>	<b>3</b>	<b>14</b>	<b>15</b>	<b>17</b>
<b>Equal</b>	<b>0.830</b>	<b>0.935</b>	<b>0.934</b>	<b>0.910</b>	<b>0.999</b>	<b>0.917</b>	<b>0.860</b>	<b>0.847</b>
<b>rank</b>	<b>17</b>	<b>7</b>	<b>8</b>	<b>11</b>	<b>2</b>	<b>10</b>	<b>14</b>	<b>15</b>

**A.6 (FMS Data) Comparison of C/R and W/B weight restrictions in 6 cases**

	W/B weight restriction						C/R weight restriction						
	X(1)	X(2)	Y(1)	Y(2)	Y(3)	Y(4)	$v_1/v_2$	$\mu_1/\mu_2$	$\mu_1/\mu_3$	$\mu_1/\mu_4$	$\mu_2/\mu_3$	$\mu_2/\mu_4$	$\mu_3/\mu_4$
<b>Case 1</b>	0.6	0.4	0.3	0.2	0.3	0.2	1.50	1.50	1.00	1.50	0.67	1.00	1.50
<b>Case 2</b>	0.2	0.8	0.1	0.4	0.2	0.3	0.25	0.25	0.50	0.33	2.00	1.33	0.67
<b>Case 3</b>	0.85	0.15	0.59	0.10	0.05	0.26	5.73	6.14	11.17	2.23	1.82	0.36	0.20
<b>Case 4</b>	0.5	0.5	0.25	0.25	0.25	0.25	1.00	1.00	1.00	1.00	1.00	1.00	1.00
<b>Case 5</b>	0.95	0.05	0.85	0.05	0.05	0.05	19.00	17.00	17.00	17.00	1.00	1.00	1.00
<b>Case 6</b>	0.95	0.05	0.45	0.05	0.05	0.45	19.00	9.00	9.00	1.00	1.00	0.11	0.11

DMU	C/R(1)	rank	W/B(1)	rank	C/R(2)	rank	W/B(2)	rank	C/R(3)	rank	W/B(3)	rank
1	0.907	4	0.926	3	0.929	3	0.981	4	0.941	4	0.998	2
2	0.879	5	0.917	4	0.919	4	0.998	2	0.914	8	0.982	3
3	0.908	3	0.876	5	0.803	5	0.776	5	0.916	7	0.943	7
4	0.973	2	0.962	2	1.000	1	1.000	1	0.938	5	0.961	5
5	1.000	1	1.000	1	0.990	2	0.983	3	0.950	3	0.981	4
6	0.814	9	0.759	9	0.571	11	0.551	10	0.967	2	0.953	6
7	0.868	6	0.814	6	0.626	9	0.603	9	1.000	1	1.000	1
8	0.853	7	0.760	8	0.747	7	0.692	8	0.922	6	0.941	8
9	0.535	12	0.046	12	0.361	12	0.035	12	0.775	11	0.230	12
10	0.754	11	0.636	11	0.621	10	0.541	11	0.812	10	0.797	10
11	0.834	8	0.782	7	0.756	6	0.740	6	0.908	9	0.928	9
12	0.763	10	0.756	10	0.725	8	0.721	7	0.757	12	0.786	11

DMU	C/R(4)	rank	W/B(4)	rank	C/R(5)	rank	W/B(5)	rank	C/R(6)	rank	W/B(6)	rank
1	0.907	3	0.949	3	1.000	1	1.000	1	0.775	8	0.816	8
2	0.883	5	0.948	4	0.961	3	0.964	3	0.761	9	0.823	7
3	0.888	4	0.854	5	0.926	7	0.918	6	0.795	7	0.831	6
4	0.986	2	0.980	2	0.895	9	0.884	8	0.829	4	0.901	3
5	1.000	1	1.000	1	0.939	5	0.931	5	0.816	5	0.875	4
6	0.769	9	0.723	10	0.897	8	0.874	9	0.982	2	0.989	2
7	0.822	7	0.773	7	0.958	4	0.940	4	0.974	3	1.000	1
8	0.839	6	0.753	8	0.930	6	0.917	7	0.814	6	0.847	5
9	0.515	12	0.048	12	0.545	12	0.234	12	1.000	1	0.277	12
10	0.738	11	0.624	11	0.771	10	0.745	11	0.753	11	0.769	9
11	0.817	8	0.778	6	0.976	2	0.967	2	0.758	10	0.768	10
12	0.757	10	0.751	9	0.761	11	0.757	10	0.653	12	0.696	11

**A.7 (Car Selection Data) Comparison of C/R and W/B weight restrictions in 6 cases**

	W/B weight restriction						C/R weight restriction						
	X(1)	X(2)	X(3)	X(4)	Y(1)	Y(2)	$v_1/v_2$	$v_1/v_3$	$v_1/v_4$	$v_2/v_3$	$v_2/v_4$	$v_3/v_4$	$\mu_1/\mu_2$
<b>Case 1</b>	0.61	0.21	0.06	0.13	0.28	0.72	2.91	10.69	4.63	3.67	1.59	0.43	0.39
<b>Case 2</b>	0.13	0.06	0.21	0.61	0.72	0.28	2.31	0.63	0.22	0.27	0.09	0.34	2.57
<b>Case 3</b>	0.30	0.30	0.15	0.25	0.36	0.64	1.03	1.99	1.23	1.94	1.20	0.62	0.57
<b>Case 4</b>	0.15	0.30	0.30	0.39	0.64	0.36	0.52	0.50	0.40	0.97	0.77	0.79	1.76
<b>Case 5</b>	0.06	0.61	0.21	0.13	0.28	0.72	0.09	0.27	0.43	2.91	4.63	1.59	0.39
<b>Case 6</b>	0.37	0.23	0.26	0.14	0.88	0.12	1.65	1.42	2.72	0.86	1.65	1.92	7.55

DMU	C/R(1)	rank	W/B(1)	rank	C/R(2)	rank	W/B(2)	rank	C/R(3)	rank	W/B(3)	rank
1	0.923	7	0.948	5	0.568	19	0.580	19	0.910	13	0.970	7
2	0.808	18	0.828	19	0.865	9	0.870	8	0.896	14	0.897	16
3	0.857	14	0.897	11	0.951	3	0.961	3	0.994	2	1.000	1
4	0.799	20	0.796	20	0.597	18	0.604	18	0.780	19	0.792	19
5	0.804	19	0.833	18	0.837	11	0.839	10	0.917	11	0.918	13
6	0.880	10	0.909	9	0.923	5	0.930	4	1.000	1	1.000	1
7	1.000	1	1.000	1	0.782	15	0.795	15	0.981	4	1.000	1
8	0.945	4	0.948	6	0.874	8	0.875	7	0.967	7	0.974	5
9	0.837	17	0.845	16	0.929	4	0.716	17	0.880	15	0.891	17
10	0.897	9	0.901	10	1.000	1	1.000	1	0.971	6	0.963	8
11	0.878	12	0.872	13	0.448	20	0.458	20	0.757	20	0.769	20
12	0.959	3	0.956	3	0.825	13	0.818	13	0.988	3	0.985	4
13	0.839	16	0.839	17	0.732	17	0.732	16	0.852	17	0.855	18
14	0.929	6	0.931	7	0.842	10	0.844	9	0.949	8	0.953	10
15	0.852	15	0.859	15	0.907	6	0.908	5	0.931	10	0.923	12
16	0.878	11	0.883	12	0.828	12	0.824	12	0.934	9	0.930	11
17	0.908	8	0.914	8	0.970	2	0.974	2	0.978	5	0.972	6
18	0.860	13	0.865	14	0.907	7	0.907	6	0.912	12	0.910	14
19	0.971	2	1.000	1	0.795	14	0.835	11	0.864	16	0.954	9
20	0.940	5	0.951	4	0.761	16	0.797	14	0.851	18	0.908	15

**A.7 (Continued) Comparison of C/R and W/B weight restrictions In 6 cases**

<b>DMU</b>	<b>C/R(4)</b>	<b>rank</b>	<b>W/B(4)</b>	<b>rank</b>	<b>C/R(5)</b>	<b>rank</b>	<b>W/B(5)</b>	<b>rank</b>	<b>C/R(6)</b>	<b>rank</b>	<b>W/B(6)</b>	<b>rank</b>
1	0.721	18	0.796	17	1.000	1	1.000	1	0.687	19	0.751	18
2	0.888	11	0.888	12	0.818	14	0.841	11	0.829	14	0.835	13
3	1.000	1	1.000	1	0.976	2	1.000	1	0.895	10	0.917	7
4	0.679	19	0.688	19	0.712	18	0.713	17	0.690	18	0.693	19
5	0.892	8	0.889	11	0.905	5	0.916	4	0.792	15	0.807	15
6	0.989	2	0.986	2	0.976	3	0.994	3	0.906	8	0.921	6
7	0.871	13	0.899	8	0.895	6	0.868	7	0.911	6	0.939	5
8	0.896	7	0.904	6	0.837	10	0.845	9	0.885	11	0.891	11
9	0.798	14	0.814	16	0.860	7	0.834	14	0.775	17	0.795	16
10	0.987	4	0.979	4	0.832	12	0.841	12	0.994	2	0.994	3
11	0.559	20	0.580	20	0.721	17	0.698	18	0.565	20	0.601	20
12	0.890	9	0.889	10	0.906	4	0.901	5	0.856	12	0.856	12
13	0.783	17	0.786	18	0.754	16	0.759	16	0.787	16	0.787	17
14	0.890	10	0.892	9	0.828	13	0.843	10	0.902	9	0.901	10
15	0.929	5	0.921	5	0.836	11	0.836	13	0.913	5	0.915	8
16	0.875	12	0.873	13	0.858	8	0.860	8	0.835	13	0.835	14
17	0.988	3	0.983	3	0.849	9	0.868	6	1.000	1	1.000	1
18	0.905	6	0.902	7	0.779	15	0.794	15	0.911	7	0.911	9
19	0.795	15	0.848	14	0.641	20	0.690	19	0.920	4	1.000	1
20	0.790	16	0.829	15	0.658	19	0.680	20	0.925	3	0.993	4

### A.8 Results of each DMU's projection points and ratio of norms of each projection

Let  $W$  be a finite-dimensional subspace of the inner product space  $V$ .

Let  $\{u_1, u_2, \dots, u_k\}$  be an orthonormal basis for  $W$  and let  $x$  be a vector in  $V$ .

The projection of  $x$  onto  $W$ , denoted  $proj_W x$  is defined to be (5.2) [18]

$$proj_W x = (x \cdot u_1)u_1 + \dots + (x \cdot u_k)u_k \quad (5.2)$$

(Calculation Procedure)

*Step 1.* Find a set of basis vectors  $v$  and an orthonormal basis  $u$  for the weight vector  $W$ .

*Step 2.* Find the projection point of each DMU according to the above equation (5.2).

*Step 3.* Calculate the norms of each projection and find the efficiency score by the equation of property (5.1).

**Case :** Two outputs have the same weights, i.e.  $\mu_1 = \mu_2$

*Step 1.*  $(\mu_1, \mu_2) = (\mu_1, \mu_1) = \mu_1 (1, 1)$ , therefore  $(1, 1)$  forms a basis  $v$  for  $W$ .

$$\|v\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\text{And orthonormal basis for } W \text{ is } u = \frac{v}{\|v\|} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$$

*Step 2.*  $proj_W x = (x \cdot u)u$

$$\text{For DMU 1, } x \cdot u = (1, 8) \cdot \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{2} + 4\sqrt{2} = 4.5\sqrt{2}$$

$$proj_W x = (x \cdot u)u = 4.5\sqrt{2} \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) = (4.5, 4.5)$$

By applying the same procedure, we can get the projection point of each DMU.

The results for each DMU are DMU 1 : (4.5, 4.5), DMU 2 : (4.5, 4.5), DMU 3 : (3.5,

3.5), DMU 4 : (4.0, 4.0), DMU 5 : (4.25, 4.25), DMU 6 : (4.0, 4.0)

Step 3. The norm of each projection vector  $\|proj_w \mathbf{x}\|$  is

$$\|proj_w \mathbf{x}\|_1 = \sqrt{4.5^2 + 4.5^2} = 6.36396, \quad \|proj_w \mathbf{x}\|_2 = \sqrt{4.5^2 + 4.5^2} = 6.36396$$

$$\|proj_w \mathbf{x}\|_3 = \sqrt{3.5^2 + 3.5^2} = 4.94975, \quad \|proj_w \mathbf{x}\|_4 = \sqrt{4.0^2 + 4.0^2} = 5.65685$$

$$\|proj_w \mathbf{x}\|_5 = \sqrt{4.25^2 + 4.25^2} = 6.0104, \quad \|proj_w \mathbf{x}\|_6 = \sqrt{4.0^2 + 4.0^2} = 5.65685$$

Therefore the efficiency score of each DMU can be calculated by the suggested property and we can see that all DMU's efficiency scores are the same as its CCR efficiency score except DMU 3 ( Because DMU 3 takes the weight vector  $\mu_1 = 2\mu_2$  )

$$\text{DMU 1 and DMU 2 : } \frac{6.36395}{6.36395} = 1, \quad \text{DMU 3 : } \frac{4.94975}{6.36395} = 0.7777$$

$$\text{DMU 4 : } \frac{5.65685}{6.36395} = 0.8888, \quad \text{DMU 5 : } \frac{6.01041}{6.36395} = 0.9444,$$

$$\text{DMU 6 : } \frac{5.65685}{6.36395} = 0.8888$$

The above results are exactly the same when we solve the CCR model with added constraint  $\mu_1 = \mu_2$  following the procedure given by (5.3). For the case of DMU 3, when we solve the CCR model with added constraint  $\mu_1 = \mu_2$ , the efficiency score is 0.7777 with  $\mu_1 = \mu_2 = 0.1111$ . But when we change the added constraint from  $\mu_1 = \mu_2$  to  $\mu_1 = 2\mu_2$ , the efficiency score will also change to be 1.00 with  $\mu_1 = 0.1538$ ,  $\mu_2 = 0.0769$ .

### A.9 Comparison between C/R and W/B weight restriction

The following Figure A-1 shows the difference of C/R and W/B weight restriction more clearly in one-input, two-output case. There are 3 DMUs of A, B and C which produce two outputs  $(y_1, y_2)$  like A(5, 5), B(2, 7), C(7, 2) using one input, and the decision maker's preference (importance) between two outputs is given to be equal. When we apply the C/R weight restriction  $\mu_1 = \mu_2$  on the above preference, Every DMU located to the right side of *bold line* will dominate the DMU A. DMUs located under the bold line don't affect to the efficiency score of the DMUs located upper side of the bold line.

On the other hand when we apply the W/B weight restriction ( $\alpha : \beta = 0.5 : 0.5$ ) in (6.2), Every DMU located to the right side of *dotted curve* will dominate the DMU A. DMUs located under the dotted curve do affect to the efficiency score of the DMUs located upper side of the bold line. That is, DMU (1, 40) takes the projection weight vector  $\mu_1 = 40\mu_2$  and its W/B efficiency score 0.2836 is measured compared with DMU C (7, 2). DMU (40, 1) takes the projection weight vector  $\mu_2 = 40\mu_1$  and its W/B efficiency score 0.2836 is measured compared with DMU B (2, 7). Therefore, if we have different DMUs (not DMU B and C) in the sample, W/B efficiency score will also become different.

The followings are results of W/B efficiency score of each DMU compared with that of DMU A, where  $[(y_1, y_2), \text{W/B efficiency score}]$ , DMU A (W/B efficiency score)

- [(1, 40), 0.2836] dominates DMU A (0.2439), but (1, 39) doesn't
- [(2, 17), 0.5528] dominates DMU A (0.5263), but (2, 16) doesn't
- [(3, 10), 0.7895] dominates DMU A (0.7692), but (3, 9) doesn't
- [(4, 7), 0.9874] dominates DMU A (0.9091), but (4, 6) doesn't



Each DMU's W/B efficiency score in this case shows the characteristics of W/B weight restrictions well.  $[(1, 40), 0.2836] < [(2, 17), 0.5528] < [(3, 10), 0.7895] < [(4, 7), 0.9874]$

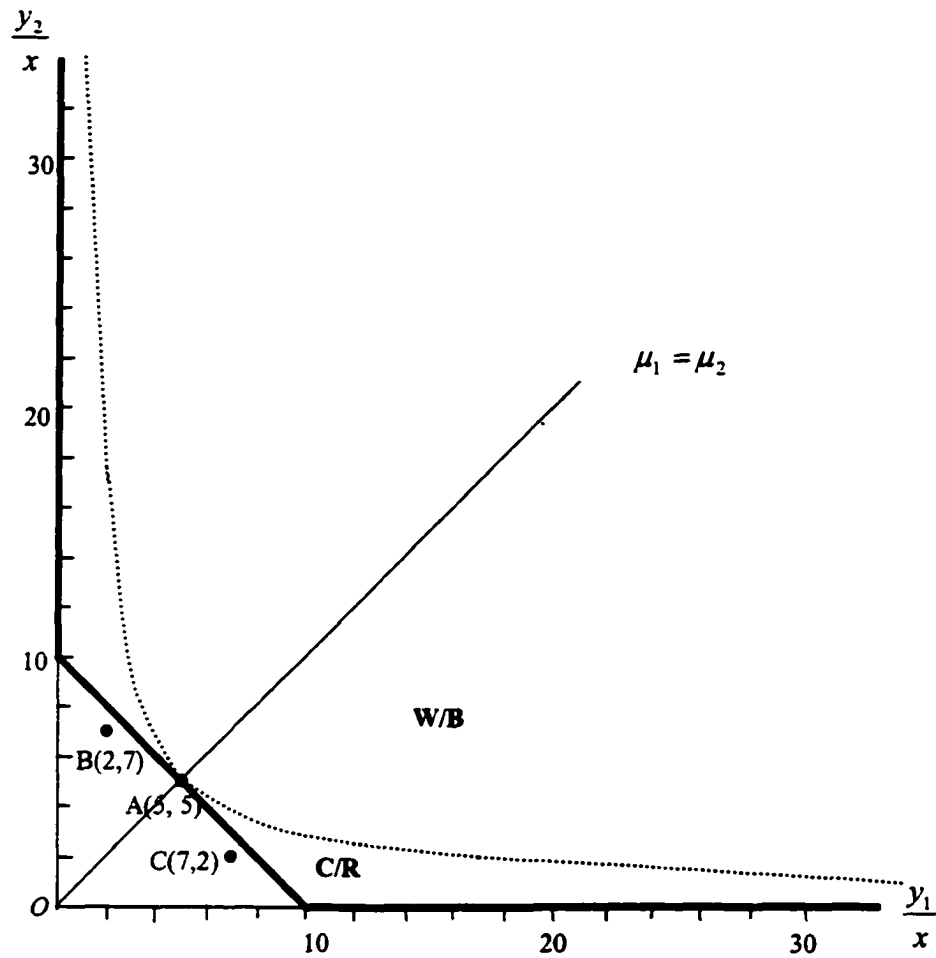


Figure A.1 Comparison between C/R and W/B weight restriction

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